Appendix A

Solutions to Selected Problems

Solutions to Chapter 3 Problems

Problem 1. Let \( f(\gamma), \ -1 \leq \gamma \leq 1 \), be a function of one variable, \( \Omega = (\theta, \varphi) \) and \( \Omega_L = (\theta_L, \varphi_L) \) are two unit vectors. Show that

\[
\int_{2\pi} f(\Omega_L \cdot \Omega) \, d\Omega = 4\varphi q(\sin \theta_L).
\]

Here \( \Omega \cdot \Omega_L \) is the scalar product of two vectors, and \( q(x), \ 0 \leq x \leq 1 \), is a function of one variable defined as

\[
q(x) = \frac{1}{2} \int_{x}^{1} f(\gamma) \, d\gamma + \frac{1}{2} \int_{0}^{x} [\alpha(\gamma, x) f(\tilde{a}) + \tilde{a}(\tilde{a}, x) f(-\tilde{a})] \, d\tilde{a},
\]

where \( \alpha(\gamma, x) + \beta(\gamma, x) = 1 \), and

\[
\tilde{a}(\tilde{a}, x) = \frac{1}{\varphi} \arccos \frac{\tilde{a} \sqrt{1 - x^2}}{x \sqrt{1 - \tilde{a}^2}}.
\]

Solution. Let

\[
g(\Omega_L) = \int_{2\pi} f(\Omega_L \cdot \Omega) \, d\Omega.
\]

In the following derivations we assume that \( \varphi_L = 0 \), that is, vector \( \Omega_L \) belongs to ZX plane. This assumption does not limit the generality of the derivations as one always can substitute \( \varphi \) with \( (\varphi - \varphi_L) \) without changing the domain of the integration (i.e., the upper hemisphere). For convenience of derivations let us transform the original system of coordinates to a new one, where \( \Omega_L \) is aligned with Z-direction (Fig. 1). This can be accomplished with rotation by angle \( -\theta_L \) around the Y-axis as specified with the following linear transform,
\[ R' = \hat{A}(-\theta_L) R, \]

where

\[
\hat{A}(-\theta_L) = \begin{bmatrix}
\cos \theta_L & 0 & -\sin \theta_L \\
0 & 1 & 0 \\
\sin \theta_L & 0 & \cos \theta_L
\end{bmatrix}
\]

and \( \vec{R} = (x, y, z) \) and \( \vec{R}' = (x', y', z') \) are vectors in the original and new coordinate system, respectively. Note the minus sign for angle \( \theta_L \). This is due to the fact that positive angles are counted outward from \( Z \)-direction. One example of transformation is for the \( Z \)-axis, which is transformed from \( Z = (0, 0, 1) \) in the original system into \( Z' = (-\sin \theta_L, 0, \cos \theta_L) \) in the new system.

To determine the integration domain in the new coordinate system, let \( \vec{v} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \) be a unit vector in the new system. The \( \vec{v} \) vector falls into the upper hemisphere of the original system if and only if

\[
\vec{v} \cdot \vec{Z}' = -\sin \theta_L \cdot \sin \theta \cos \varphi + 0 \cdot \sin \theta \sin \varphi + \cos \theta_L \cdot \cos \theta \geq 0.
\]

Therefore,

\[
\cos \varphi \leq \frac{\cos \theta_L \cos \theta}{\sin \theta_L \sin \theta}.
\]

Figure. 1 Coordinate system transform via rotation by angle \(-\theta_L\) around the Y-axis.

Note that the angular domain of the upper hemisphere in the original coordinate system is \( \theta \in [0, \pi] \) and \( \varphi \in [0, 2\pi] \). This domain is specified differently in the new coordinate system as shown in Fig. 2. Namely, three sectors with respect to \( \theta \) can be identified: \( \theta \in [0, \pi/2 - \theta_L] \) (shown in Red), \( \theta \in [\pi/2 - \theta_L, \pi/2 + \theta_L] \) (shown in green), and \( \theta \in [\pi/2 + \theta_L, \pi] \) (shown in blue). The corresponding variations of \( \varphi \) can be derived from the algebraic analysis of Eq. (2). The final expression for the angular domain in the new coordinate system is:
\[
\begin{align*}
\cos \theta_L \cos \theta & > 1, \quad \theta \in [0, \frac{\pi}{2} - \theta_L), \\
\cos \theta_L \cos \theta & \leq 1, \quad \theta \in [\frac{\pi}{2} - \theta_L, \frac{\pi}{2} + \theta_L], \\
\cos \theta_L \cos \theta & < -1, \quad \theta \in (\frac{\pi}{2} + \theta_L, \pi], \\
\phi & \in [\arccos \frac{\cos \theta_L \cos \theta}{\sin \theta_L \sin \theta}, 2\pi - \arccos \frac{\cos \theta_L \cos \theta}{\sin \theta_L \sin \theta}].
\end{align*}
\]  

\( (3) \)

In a view that in the new coordinate system the direction \( \mathbf{L} \) coincides with \( Z' \), the argument of function \( f(\mathbf{L} \cdot \mathbf{\Omega}) \) in Eq. (1) simplifies, i.e., \( \mathbf{L} \cdot \mathbf{\Omega} = \cos \theta \). Therefore,

\[
g(\mathbf{L}) = \int f(\mathbf{L} \cdot \mathbf{\Omega}) d\mathbf{\Omega} = \int f(\cos \theta) d\mathbf{\Omega}
\]

\[
= \int_0^{\frac{\pi}{2} - \theta_L} f(\cos \theta)(- \cos \phi) d\phi + \int_{\frac{\pi}{2} + \theta_L}^{\pi} f(\cos \theta)(- \cos \phi) d\phi
\]

\[
= 2\pi \int_0^{\theta_L} f(\sin \theta) d(\sin \theta)
\]

\[
+ \int_{\theta_L}^{\frac{\pi}{2}} f(\sin \theta) \left[2\pi - 2 \arccos \left( \frac{\cos \theta \cos \phi}{\sin \theta \sin \phi} \right)\right] d(\sin \theta)
\]

Figure. 2 The upper hemisphere of the original system of coordinates XYZ in a new system of coordinates \( X'Y'Z' \). Vector \( \mathbf{L} \) is aligned with axis \( Z' \) of the new system.
Note that in the last step of derivations we substituted $\theta$ with $\pi/2 - \theta$, and accounted for the fact that $\cos \theta = \sin(\pi/2 - \theta)$, $\sin \theta = \cos(\pi/2 - \theta)$, and $\arccos(-\psi) = \pi - \arccos(\psi)$. Finally, let $x = \sin(\theta_L)$, $\gamma = \sin(\theta)$, and

$$\hat{a}(\hat{a}, x) = \frac{1}{\hat{a}} \arccos \left( \frac{\cos \theta \sin \gamma}{\sin \theta \cos \gamma} \right) = \frac{1}{\hat{a}} \arccos \frac{\hat{a} \sqrt{1 - x^2}}{\sqrt{1 - \hat{a}^2}},$$

$$\alpha(\gamma, x) = 1 - \beta(\gamma, x).$$

Substituting the new variables in Eq. (4) and changing the limits of integration according to the variable $\gamma$, we finally have

$$g(\Omega_L) = 2\pi \int_{\hat{a}} (\hat{f}(\gamma) d\hat{a} + 2\pi \int_{\hat{a}} (\alpha(\gamma, x) f(\hat{a}) + \hat{a}(\hat{a}, x) f(-\hat{a})) d\hat{a} = 4\pi q(x).$$

**Problem 2.** Letting $f(\gamma) = \hat{f}$, show that

$$\int_{2\pi} \Omega \cdot \Omega | d\Omega = \pi.$$

**Solution.** According to the results of Problem 1

$$\int_{2\pi} \Omega \cdot \Omega | d\Omega = \int_{2\pi} (\gamma) d\Omega = 4\pi \frac{1}{2} \left[ \int_{x}^{z} \gamma d\gamma + \int_{0}^{\alpha(\gamma, x) + \beta(\gamma, x)} \gamma d\gamma \right].$$

Taking into account that $\alpha(\lambda, x) + \beta(\lambda, x) = 1$ (cf. Problem 1), we have

$$\int_{2\pi} \Omega \cdot \Omega | d\Omega = 2\pi \left[ \int_{x}^{z} \gamma d\gamma + \int_{0}^{\gamma} \gamma d\gamma \right] = 2\pi \int_{0}^{\gamma} d\gamma = \pi.$$

**Problem 3.** Show that the leaf albedo for the bi-Lambertian model is

$$\int d\Omega \hat{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \hat{n}_{L,d} + \hat{\sigma}_{L,d}.$$
**Solution.** The bi-Lambertian model for diffuse leaf scattering phase function is (cf. Chapter 3, Eq. (9))

$$\tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \begin{cases} \frac{1}{2} \tilde{a}_{L,d} |\Omega \cdot \Omega_L|, & (\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) < 0, \\ \frac{1}{2} \delta_{L,d} |\Omega \cdot \Omega_L|, & (\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) > 0. \end{cases} \tag{1}$$

Note that the angles $\Omega'$ and $\Omega_L$ are fixed in the integral of interests and integration over whole sphere can be spitted into two parts, namely,

$$\int_{4\pi} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) > 0} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) + \int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) < 0} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L). \tag{2}$$

Taking into account Eq. (1), we have

$$\int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) > 0} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \frac{1}{\pi} \tau_{L,d} \int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) > 0} |\Omega \cdot \Omega_L|, \tag{3a}$$

$$\int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) < 0} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \frac{1}{\pi} \rho_{L,d} \int_{(\Omega \cdot \Omega_L)(\Omega' \cdot \Omega_L) < 0} |\Omega \cdot \Omega_L|. \tag{3b}$$

In view that the angles $\Omega'$ and $\Omega_L$ are fixed, the integrals on the right hand side of Eqs. (3a) and (3b) are identical and equal to the integral over hemisphere ($2\pi^+$ or $2\pi^-$). The last integral was evaluated in the Problem 2, namely,

$$\int_{2\pi^+} |\Omega \cdot \Omega_L| = \pi. \tag{4}$$

Combining Eqs. (2)-(4) we finally have

$$\int_{4\pi} \tilde{a}_{L,d}(\Omega' \rightarrow \Omega, \Omega_L) = \frac{1}{\pi} \tau_{L,d} \pi + \frac{1}{\pi} \rho_{L,d} \pi = \tau_{L,d} + \rho_{L,d}. \tag{5}$$

**Problem 5.** Prove Equation (14).

**Solution.** Equation (14) in Chapter 3 states:

$$\frac{1}{4\pi} \int_{2\pi^+} G(r, \Omega) = \frac{1}{2}, \tag{1}$$
where (cf. Chapter 3, Eq. (2))

\[
G(r, \Omega) = \frac{1}{2\pi} \int_{2\pi} d\Omega_L \, g_L (\Omega_L) |\Omega_L \cdot \Omega|,
\]

\[
\frac{1}{2\pi} \int_{2\pi} d\Omega_L \, g_L (\Omega_L) = 1.
\]

Combining Eq. (1)-(2) and result of Problem 2, we have

\[
\frac{1}{2\pi} \int d\Omega \, G(r, \Omega) = \frac{1}{2\pi} \int \frac{1}{2\pi} \int d\Omega_L \, g_L (\Omega_L) |\Omega_L \cdot \Omega|
\]

\[
= \frac{1}{4\pi^2} \int d\Omega_L \, g_L (\Omega_L) \int d\Omega |\Omega_L \cdot \Omega|
\]

\[
= \frac{1}{4\pi^2} 2\pi \cdot \pi = \frac{1}{2}.
\]

**Problem 6.** Show that the geometry factor \( G(r, \Omega) \) for spherically distributed leaf normals depends neither \( r \) nor \( \Omega \) and is equal to 0.5.

**Solution.** Recall (cf. Chapter 3, Eq. (2)),

\[
G(r, \Omega) = \frac{1}{2\pi} \int_{2\pi} d\Omega_L \, g_L (\Omega_L) |\Omega_L \cdot \Omega|
\]

Recall also, that for spherically distributed leaf normals \( g_L (\Omega_L) = 1 \) (cf. Chapter 3, Eq. (5f)). Combining this result with the result of the Problem 2, we have

\[
G(r, \Omega) = \frac{1}{2\pi} \int_{2\pi} d\Omega_L |\Omega_L \cdot \Omega| = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}.
\]

**Problem 7.** Show that \( G = \mu \) for horizontal leaves, and \( G = (2 / \pi) \sin \theta \) for vertical leaves.

**Solution.** Recall (cf. Chapter 3, Eq. (2)),

\[
G(r, \Omega) = \frac{1}{2\pi} \int_{2\pi} d\Omega_L \, g_L (\Omega_L) |\Omega_L \cdot \Omega|
\]
Further, let $\Omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ and $\Omega_L = (\sin \theta_L \cos \varphi_L, \sin \theta_L \sin \varphi_L, \cos \theta_L)$. Therefore,

$$\Omega_L \cdot \Omega = (\sin \theta \cos \varphi_L) (\sin \theta \cos \varphi) + (\sin \theta \sin \varphi_L) (\sin \theta \sin \varphi) + (\cos \theta_L) (\cos \theta)$$

$$= \sin \theta \sin \theta_L \cos(\varphi_L - \varphi) + \cos \theta_L \cos \theta.$$

For horizontal leaves the derivations for $G(r, \Omega)$ are as follows. The probability density of leaf normal orientation (cf. Chapter 3, Eq. (4)) is

$$g_L(\Omega_L) = \frac{\delta(\theta_L - 0)}{\sin \theta_L}.$$

Therefore,

$$G(r, \Omega) = \frac{1}{2\pi} \int_0^{\pi/2} \sin \theta_L d\theta_L \int_0^{2\pi} d\varphi_L \ g_L(\Omega_L) \left| \Omega_L \cdot \Omega \right|$$

$$= \frac{1}{2\pi} \int_0^{\pi/2} d\theta_L \int_0^{2\pi} d\varphi_L \sin \theta_L \frac{\delta(\theta_L - 0)}{\sin \theta_L} \left| \sin \theta_L \sin \theta \cos(\varphi_L - \varphi) + \cos \theta_L \cos \theta \right|$$

$$= \left| \cos \theta \right| = \mu.$$

Similar derivations can be performed for vertical leaves. In this case the probability density of leaf normal orientation is

$$g_L(\Omega_L) = \frac{\delta(\theta_L - \pi/2)}{\sin \theta_L}.$$

Therefore,

$$G(r, \Omega) = \frac{1}{2\pi} \int_0^{\pi/2} \sin \theta_L d\theta_L \int_0^{2\pi} d\varphi_L \sin \theta_L \frac{\delta(\theta_L - \pi/2)}{\sin \theta_L} \left| \sin \theta_L \sin \theta \cos(\varphi_L - \varphi) + \cos \theta_L \cos \theta \right|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi_L \left| \sin \theta \cdot \cos(\varphi_L - \varphi) + 0 \right| = \frac{1}{\pi} \left| \sin \theta \right|.$$
**Problem 8.** Let the polar angle, $\theta_L$, and azimuth, $\varphi_L$, of leaf normals be independent (see Eq. 3). Show that

$$ G(\bar{r}, \mu) = \int_0^1 d\mu_L \bar{g}_L(\bar{r}, \mu_L) \psi(\mu, \mu_L), $$

where $\mu_L = \cos \theta_L$, $\mu = \cos \theta$, and

$$ \phi(i, i_L) = \frac{1}{2\pi} \int_0^{2\pi} h_L(\bar{\varphi}_L) |\Omega \cdot \Omega_L| d\bar{\varphi}_L. $$

**Solution.** Since the polar angle, $\theta_L$, and azimuth angle, $\varphi_L$, of leaf normals are independant, the following representation of the probability density of leaf normal orientation is valid (cf. Chapter 3, Eq. (3)):

$$ g_L(\Omega_L) = \bar{g}_L(i_L) h_L(\bar{\varphi}_L), $$

where $\bar{g}_L(i_L)$ and $h_L(\bar{\varphi}_L)/2\delta$ are the probability density functions of leaf normal inclination and azimuth, respectively. Therefore,

$$ G(\bar{r}, \mu_L) = \frac{1}{2\pi} \int_{2\pi} d\Omega_L g_L(\bar{r}, \Omega_L) |\Omega_L \cdot \Omega| $$

$$ = \frac{1}{2\pi} \int_0^1 d\mu_L \int_{2\pi} d\varphi_L \bar{g}_L(\bar{r}, \mu_L) h_L(\varphi_L) |\Omega_L \cdot \Omega| $$

$$ = \frac{1}{2\pi} \int_0^1 d\mu_L \bar{g}_L(\bar{r}, \mu_L) \int_{2\pi} d\varphi_L h_L(\varphi_L) |\Omega_L \cdot \Omega| $$

$$ = \frac{1}{2\pi} \int_0^1 d\mu_L \bar{g}_L(\bar{r}, \mu_L) \psi(\mu, \mu_L). $$

**Problem 9.** Show that in canopies where leaf normals are distributed uniformly along the azimuthal coordinate [i.e., $h_L(\varphi_L) = 1$], $\psi(\mu, \mu_L)$ can be reduced to

$$ \phi(i, i_L) = \begin{cases} \| i \cdot i_L \| \; & \text{if} \; \| i \cdot i_L \| \geq \| \sin \varphi \sin \varphi_L \|, \\
(2 \varphi_i/\delta - 1) + (2/\delta) \sqrt{1 - i^2} \sqrt{1 - i^2} \sin \varphi_i, & \text{otherwise}, \end{cases} $$

where the branch angle $\varphi_i$ is $\arccos(-\cot \theta \times \cot \theta_L)$. 


Solution. Recall (cf. Problem 6),

\[ \mathbf{\Omega}_L \cdot \mathbf{\Omega} = \sin \theta_L \sin \theta \cos (\varphi_L - \varphi) + \cos \theta_L \cos \theta. \]

Therefore,

\[
\psi(\mu, \mu_L) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi_L h_L(\varphi_L) \left| \mathbf{\Omega}_L \cdot \mathbf{\Omega} \right|
= \frac{1}{2\pi} \int_0^{2\pi} d\varphi_L \left| \sin \theta_L \sin \theta \cos (\varphi_L - \varphi) + \cos \theta_L \cos \theta \right|
= \frac{1}{\pi} \int_0^\pi d\varphi_L \left| \sin \theta_L \sin \theta \cos (\varphi_L - \varphi) + \cos \theta_L \cos \theta \right|
= \frac{1}{\pi} \int_0^\pi d\varphi_L \left| \sin \theta_L \sin \theta \cos (\varphi_L + \cot \theta_L \cot \theta) \right|
\]

(1)

Note that in the above derivations we took into account that \( h_L(\varphi_L) = 1 \) and also symmetry of function \((a \cdot \cos(\xi) + b)\) in the interval \(\xi \in [0, 2\pi]\). In the derivations to follow we need to consider two cases. First, consider the case, when \( |\cot \theta_L \cot \theta| \geq 1 \). In this case the expression \((\cos \varphi_L + \cot \theta_L \cot \theta)\) in the Eq. (1) does not change the sign over the whole interval of integration, \([0, \pi]\). Therefore,

\[
\phi(i, i_L) = \frac{1}{\pi} \int_0^\pi d\varphi_L \left| \sin \theta_L \sin \theta \cos (\varphi_L + \cot \theta_L \cot \theta) \right|
= \frac{1}{\pi} \int_0^\pi d\varphi_L \left| \sin \theta_L \sin \theta \cos (\varphi_L + \cot \theta_L \cot \theta) \right|
= |\cos \theta_L \cos \theta|
\]

(2a)

Now consider the second case, when \(|\cot \theta_L \cot \theta| < 1\). Here the value of the expression \((\cos \varphi_L + \cot \theta_L \cot \theta)\) in the Eq. (1) is monotonically decreasing in the interval \(\varphi \in [0, \pi]\). The value of \(\varphi = \varphi^*\), where the integrand changes the sign is given by

\[
\cos \varphi^* + \cot \theta_L \cot \theta = 0 \Rightarrow \varphi^* = \arccos \left[-\cot \theta_L \cot \theta\right]
\]

Now we can evaluate Eq. (1) by splitting the interval of integration into two parts, where the integrand has the opposite signs,
\( \varnothing(i, \varnothing_L) = \frac{1}{\pi} \int_{0}^{\pi} d\varnothing_L \left| \sin \theta_L \sin \theta \left( \cos \varnothing_L + \cot \theta_L \csc \theta \right) \right| \\
= \frac{1}{\pi} \int_{0}^{\varnothing^*} d\varnothing_L \left( \sin \theta_L \sin \theta \cos \varnothing_L + \cos \theta_L \cos \theta \right) \left[ \pi \int_{\varnothing^*}^{\pi} d\varnothing_L \left( \sin \theta_L \sin \theta \cos \varnothing_L + \cos \theta_L \cot \theta \right) \right] \\
= \frac{1}{\pi} \left[ 2 \sin \theta_L \sin \theta \sin \varnothing^* + \cos \theta_L \cos \theta \left( 2\varnothing^* - \pi \right) \right]. \quad (2b) \\

Combined, the Eq. (2a)-(2b) present the solution to the problem.

**Problem 10.** Show that in canopies with constant leaf normal inclination but uniform orientation along the azimuth [cf. Eq. (4)], \( G(\mu) = \psi(\mu, \mu_L) \)

**Solution.** For canopies with independent polar angle, \( \theta_L \), and azimuth, \( \varnothing_L \), of leaf normals (cf. Problem 7), we have

\[
G(\varnothing_L, \mu) = \int_{0}^{1} d\mu_L \bar{G}_L(\varnothing_L, \mu_L) \psi(\mu, \mu_L), \quad (1)
\]

where \( \mu_L = \cos \theta_L \), \( \mu = \cos \theta \), and

\[
\varnothing(i, \varnothing_L) = \frac{1}{2\pi} \int_{0}^{\varnothing^*} h_L(\varnothing_L) | \Omega \cdot \bar{\Omega}_L | d\varnothing_L. \quad (2)
\]

The condition of constant leaf normal inclination and uniform orientation along azimuth implies

\[
\bar{G}_L(\varnothing_L, \mu_L) = \frac{\delta(\theta_L - \theta^*)}{\sin \theta_L} = \delta(\mu_L - \mu^*), \\
h_L(\varnothing_L) = 1. \quad (3)
\]

Combining Eqs. (1)-(3), we have

\[
G(\varnothing_L, \mu) = \int_{0}^{1} d\mu_L \bar{G}_L(\varnothing_L, \mu_L) \psi(\mu, \mu_L) \\
= \int_{0}^{1} d\mu_L \delta(\mu_L - \mu^*) \psi(\mu, \mu_L), \\
= \psi(\mu, \mu^*).
\]
Problem 11. Using Eq. (6), derive the \( \phi(i, i_L) \) in the case of heliotropic orientations.

Solution. Recall (cf. Problem 7),

\[
\phi(i, i_L) = \frac{1}{2\pi} \int_{0}^{2\pi} h_L(\bar{\phi}_L) | \Omega \cdot \Omega_L | d\bar{\phi}_L. \tag{1}
\]

In the case when the leaf azimuths have a preferred orientation with respect to the solar azimuth orientation (heliotropism) the \( h_L \) function may be modeled as follows (Chapter 3, Eq. (6)),

\[
\frac{1}{2\partial} h_L(\bar{\phi}_L, \bar{\phi}) = \frac{1}{\partial} \cos^2(\bar{\phi} - \bar{\phi}_L - \varphi). \tag{2}
\]

Recall also (cf. Problem 6),

\[
\Omega_L \cdot \Omega = \sin \theta_L \sin \theta \cdot \cos(\varphi_L - \varphi) + \cos \theta_L \cos \theta. \tag{3}
\]

Combining Eq. (1)-(3), we have

\[
\phi(i, i_L) = \frac{1}{2\pi} \int_{0}^{2\pi} h_L(\bar{\phi}_L) | \Omega \cdot \Omega_L | d\bar{\phi}_L = \frac{1}{\pi} \int_{0}^{2\pi} d\varphi_L \cos^2(\varphi - \varphi_L - \eta) | \sin \theta_L \sin \theta \cos(\varphi - \varphi_L) + \cos \theta_L \cos \theta |
\]

\[
= \frac{1}{\pi} \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta) | \sin \theta_L \sin \theta \cos \alpha + \cos \theta_L \cos \theta |
\]

\[
= \frac{1}{\pi} \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta) | \sin \theta_L \sin \theta \cdot (\cos \alpha + \text{ctg} \theta_L \text{ctg} \theta) |, \tag{4}
\]

where \( \alpha = \varphi - \varphi_L \). To evaluate the above integral we need to consider two cases (cf. Problem 8). First, consider the case, when \(|\text{ctg} \theta_L \text{ctg} \theta| \geq 1 \). In this case the integrand does not change the sign over whole interval of integration. Therefore,

\[
\phi(i, i_L) = \frac{1}{\pi} \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta) | \sin \theta_L \sin \theta \cos \alpha + \cos \theta_L \cos \theta |
\]

\[
= \frac{1}{\pi} \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta) | \sin \theta_L \sin \theta \cos \alpha + \cos \theta_L \cos \theta |
\]

\[
= \frac{1}{\pi} \int_{0}^{2\pi} \sin \theta_L \sin \theta \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta) \cos \alpha + \cos \theta_L \cos \theta \int_{0}^{2\pi} d\alpha \cos^2(\alpha - \eta). \tag{5}
\]
The evaluation of two definite integrals in Eq. (5) requires derivations of corresponding indefinite integrals, $I_1$ and $I_2$:

\[
I_1 = \sin \theta_2 \sin \theta \int d\alpha \cos^2 (\alpha - \eta) \cos \alpha = \sin \theta_2 \sin \theta \int d\alpha \frac{1 + \cos(2\alpha - 2\eta)}{2} \cos \alpha
\]
\[
= \frac{\sin \theta_2 \sin \theta}{2} \left[ \sin \alpha + \frac{\sin(\alpha - 2\eta)}{2} + \frac{3\alpha - 2\eta}{6} \right].
\] (6a)

\[
I_2 = \cos \theta_2 \cos \theta \int d\alpha \cos^2 (\alpha - \eta) = \cos \theta_2 \cos \theta \int d\alpha \frac{1 + \cos(2\alpha - 2\eta)}{2}
\]
\[
= \frac{\cos \theta_2 \cos \theta}{2} \left[ \alpha + \frac{\sin(2\alpha - 2\eta)}{2} \right].
\] (6b)

Therefore, the corresponding definite integrals over interval $\alpha \in [0, 2\pi]$ are

\[
I_1 \bigg| _{\alpha=0}^{\alpha=2\pi} = 0 \quad \text{and} \quad I_2 \bigg| _{\alpha=0}^{\alpha=2\pi} = \pi \cos \theta_2 \cos \theta.
\] (7)

Combining Eqs. (5)-(7), we have

\[
\phi(i, i_{\perp}) = |\cos \theta \cos \theta_2| \equiv \mu \mu_2.
\] (8)

Now consider the second case, when $|\cot \theta_2 \cot \theta| < 1$. In this case the expression under sign of integral in Eq. (4), $(\cos \alpha + \cot \theta_2 \cot \theta)$, will change sign two times (first at $\alpha^*$ and second at $2\pi - \alpha^*$) over the interval $\alpha \in [0, 2\pi]$. The value of $\alpha^*$ is given by

\[
(\cos \alpha + \cot \theta_2 \cot \theta) = 0 \quad \Rightarrow \quad \alpha^* = \arccos \left[ -\cot \theta_2 \cot \theta \right].
\] (9)

In order to evaluate integral in Eq. (4) we need to consider three intervals, where integrand has constant sign: $\alpha \in [0, \alpha^*]$, $\alpha \in [\alpha^*, 2\pi - \alpha^*]$, and $\alpha \in [2\pi - \alpha^*, 2\pi]$. Therefore,

\[
\phi(i, i_{\perp}) = \frac{1}{\pi} \int_0^{2\pi} d\alpha \left| \sin \theta_2 \sin \theta \cos^2 (\alpha - \eta) \cos \alpha + \cos \theta_2 \cos \theta \cos^2 (\alpha - \eta) \right|
\]
\[
= \frac{1}{\pi} \left| \left[ I_1 + I_2 \right]_{\alpha=0}^{\alpha=\alpha^*} - 2 \left[ I_1 + I_2 \right]_{\alpha=2\pi-\alpha^*} + \left[ I_1 + I_2 \right]_{\alpha=2\pi} \right|
\]
\[
= \frac{1}{\pi} \left| \left[ I_1 + I_2 \right]_{\alpha=0}^{\alpha=\alpha^*} - 2 \left[ I_1 + I_2 \right]_{\alpha=2\pi-\alpha^*} \right|
\]
\[
= \frac{1}{\pi} \left| I_1_{\alpha=0}^{\alpha=\alpha^*} - 2 \cdot I_1_{\alpha=2\pi-\alpha^*} + I_2_{\alpha=2\pi} - 2 \cdot I_2_{\alpha=2\pi-\alpha^*} \right|,
\] (10)
where
\[
I_1\Big|_{\theta=0}^{2\pi} - 2 \cdot I_1\Big|_{\alpha'}^{2\pi-\alpha'} = \sin \theta \sin \theta_L \cdot \left( 2 \sin \alpha^* + \sin \alpha^* \cos (2\eta) + \frac{1}{3} \sin (3\alpha^*) \cos (2\eta) \right), \quad (11a)
\]
\[
I_2\Big|_{\theta=0}^{2\pi} - 2 \cdot I_2\Big|_{\alpha'}^{2\pi-\alpha'} = \cos \theta \cos \theta_L \cdot \left( 2\alpha^* - \pi + \sin (2\alpha^*) \cos (2\eta) \right), \quad (11b)
\]
Combining Eqs. (10)-(11), we have
\[
\phi(i, i_L) = \left| \frac{\sin \theta \sin \theta_L}{\pi} \cdot \left( 2 \sin \alpha^* + \sin \alpha^* \cos (2\eta) + \frac{1}{3} \sin (3\alpha^*) \cos (2\eta) \right) \right| + \frac{\cos \theta \cos \theta_L}{\pi} \cdot \left( 2\alpha^* - \pi + \sin (2\alpha^*) \cos (2\eta) \right), \quad (12)
\]
where \(\alpha^*\) is given by Eq. (9). Overall, Eqs. (8) and (12) present the complete solution to the problem. Finally, note that in the special case of dia-heliotropic distribution, \((\eta = 0)\), the Eq. (12) reduces to
\[
\phi(i, i_L) = \left| \frac{\sin \theta \sin \theta_L}{\pi} \cdot \left( 3 \sin \alpha^* + \frac{1}{3} \sin (3\alpha^*) \right) \right| + \frac{\cos \theta \cos \theta_L}{\pi} \cdot \left( 2\alpha^* - \pi + \sin (2\alpha^*) \right),
\]
and in the case of para-heliotropic distribution \((\eta = \pi/2)\), to
\[
\phi(i, i_L) = \left| \frac{\sin \theta \sin \theta_L}{\pi} \cdot \left( 2 \sin \alpha^* - \frac{1}{3} \sin (3\alpha^*) \right) \right| + \frac{\cos \theta \cos \theta_L}{\pi} \cdot \left( 2\alpha^* - \pi - \sin (2\alpha^*) \right).
\]

**Problem 12.** Calculate area scattering phase function for diffuse radiation, \(\Gamma_k(\Omega' \rightarrow \Omega)\), in the case of uniform leaf normal distribution, \(g_L = 1\).

**Solution.** Recall (Chapter 3, Eq. (16)-(17)),
\[
\tilde{A}_d(\Omega' \rightarrow \Omega) = \tilde{n}_{L,d} \tilde{A}_d(\Omega' \rightarrow \Omega) + \delta_{L,d} \tilde{A}_d(\Omega' \rightarrow \Omega), \quad (1a)
\]
where,
\[
\Gamma_d(\Omega' \rightarrow \Omega) = \pm \frac{1}{2\pi} \int_{0}^{1} \int_{\partial \Omega} g_L (\hat{\Omega} \cdot \hat{\Omega}_L) (\Omega' \cdot \Omega_L). \quad (1b)
\]
The $(\pm)$ in the above definition indicates that the $\varphi_L$ integration is over that portion of the interval $[0, 2\pi]$ for which the integrand is either positive $(\pm)$ or negative $(-)$. We need to calculate phase function for the case of uniform leaf normal orientation, $g_L = 1$. In view of the symmetry of the integrand with respect to $\Omega_L$, the integral over the upper hemisphere is equal to that over lower hemisphere and,

$$\Gamma^z(\Omega' \to \Omega) = \pm \frac{1}{2\partial} \int_{2\pi} d\Omega_L (\Omega \cdot \Omega_L) (\Omega' \cdot \Omega_L) = \pm \frac{1}{2} \left[ \frac{1}{2\partial} \int_{4\pi} d\Omega_L (\Omega \cdot \Omega_L) (\Omega' \cdot \Omega_L) \right].$$  \hspace{1cm} (1c)

The integrand depends on three vectors: $\Omega$, $\Omega'$, $\Omega_L$. In order to simplify integration, let us choose the coordinate system where X-Y plane coincides with the plane of vectors $\Omega'$ and $\Omega$ ($\mu' = \mu = 0$). Therefore,

$$\begin{align*}
\Omega &= (\cos \varphi, \sin \varphi, 0), \\
\Omega' &= (\cos \varphi', \sin \varphi', 0), \\
\Omega_L &= (\sin \theta_L \cos \varphi_L, \sin \theta_L \sin \varphi_L, \cos \theta_L).
\end{align*}$$  \hspace{1cm} (2)

Taking into account Eq. (2) and performing trigonometric transformations, we have

$$\Omega_L \cdot \Omega' (\Omega_L \cdot \Omega) = \sin^2 \theta_L \cos(\varphi_L - \varphi) \cos(\varphi_L - \varphi').$$ \hspace{1cm} (3)

Let $\beta = \varphi' - \varphi$, an angle between $\Omega$ and $\Omega'$, $\beta = \arccos(\Omega \cdot \Omega')$. Let $\chi = \varphi_L - \varphi'$, and $d\chi = d\varphi_L$. Therefore,

$$\Gamma^z = \pm \frac{1}{4\pi} \int_{-1}^{1} d\mu_L (1 - \mu^2) \int_{0}^{2\pi} d\chi \cos(\chi + \beta) \cos(\chi)$$

$$= \pm \frac{1}{4\pi} \int_{-1}^{1} d\mu_L (1 - \mu^2) \int_{0}^{2\pi} d\chi \frac{1}{2} \left[ \cos(\beta) + \cos(2\chi + \beta) \right]$$

$$= \pm \frac{1}{4\pi} \int_{-1}^{1} d\mu_L (1 - \mu^2) \int_{0}^{2\pi} d\chi \frac{1}{2} \left[ \cos(\beta) + \cos(\chi) \right].$$  \hspace{1cm} (4)

Note, the integrand in Eq. (4) is changing sign two times over the interval $[0; 2\pi]$:

$$\cos(y) = -\cos(\beta) \Rightarrow y = \pi \pm \beta.$$

Namely, the integrand is positive over $[0; \pi - \beta]$, negative over $[\pi - \beta; \pi + \beta]$ and again positive over $[\pi + \beta; 2\pi]$. Therefore,
\[
I^+(\beta) = \frac{1}{2} \left\{ \int_0^{\pi - \beta} [\cos(\beta) + \cos(y)] \, dy + \int_{\pi + \beta}^{2\pi} [\cos(\beta) + \cos(y)] \, dy \right\}
= \int_0^{\pi - \beta} [\cos(\beta) + \cos(y)] \, dy
= (\pi - \beta) \cos(\beta) + \sin(\beta),
\]  
\[(5a)\]

and

\[
I^- (\beta) = \frac{1}{2} \int_{\pi - \beta}^{\pi + \beta} [\cos(\beta) + \cos(y)] \, dy
= \int_{\pi - \beta}^{\pi + \beta} [\cos(\beta) + \cos(y)] \, dy
= \beta \cos(\beta) - \sin(\beta). 
\]  
\[(5b)\]

Combining Eq. (4) and (5), we have,

\[
\Gamma^z = \pm \frac{1}{4\pi} \int_1^{1^z} d\mu_L (1 - \mu^2 L) = \pm \frac{I^z (\beta)}{3\pi}. 
\]  
\[(6)\]

Finally, substituting Eq. (6) into Eq. (1), we have

\[
\hat{A}_d (\Omega' \rightarrow \Omega) = \hat{n}_{L, d} \hat{A}_d^0 (\Omega' \rightarrow \Omega) + \hat{d}_{L, d} \hat{A}_d^+ (\Omega' \rightarrow \Omega)
= \frac{1}{3\pi} \left\{ \hat{n}_{L, d} [-\beta \cos(\beta) + \sin(\beta)] + \hat{d}_{L, d} [(\pi - \beta) \cos(\beta) + \sin(\beta)] \right\}
= \frac{\omega_{L, d}}{3\pi} [\sin(\beta) - \beta \cos(\beta)] + \frac{\hat{d}_{L, d}}{3} \cos(\beta).
\]

**Problem 13.** Given the direction of incoming, \(\Omega\), and reflected, \(\Omega'\), fluxes at the leaf surface (\(\|\Omega\| = \|\Omega'\| = 1\)) show that the direction of leaf normals \(\hat{n}_L (\mu_L, \varphi_L)\) in the case of specular reflection is given by

\[
\mu_L = \frac{\mu' - \mu}{\sqrt{2(1 - \Omega' \cdot \Omega)}},
\]

\[
\tan \varphi_L = \frac{\sqrt{1 - \mu'^2} \sin \varphi' - \sqrt{1 - \mu^2} \sin \varphi}{\sqrt{1 - \mu'^2} \cos \varphi' - \sqrt{1 - \mu^2} \cos \varphi}.
\]

**Solution.** Consider the geometry of interaction of solar beams with a leaf surface in the case of specular reflection as shown in Fig. (1). Given the incoming, \(\hat{U}\), and reflected, \(\hat{U}'\), directions,
the angle between them is given by $\cos \alpha = \hat{U} \cdot \hat{U}'$. The leaf normal in the case of specular reflection is

$$\hat{n}_L = \frac{\hat{U} - \hat{U}'}{\|\hat{U} - \hat{U}'\|}.$$  

(1)

The norm $\|\hat{U} - \hat{U}'\|$ can be calculated as follows (cf. Fig. 1(b)):

$$\|\hat{U} - \hat{U}'\| = 2\sin \frac{\alpha}{2} = 2\sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{2(1 - \Omega \cdot \Omega')}.$$  

(2)

Figure. 1 The geometry of interaction of radiation with leaf. Panel (a) shows direction of leaf surface, leaf normal ($\hat{n}_L$), incoming ($\Omega$) and reflected ($\Omega'$) beams. Panel (b) is a schematic plot to evaluate the norm of $\|\hat{U} - \hat{U}'\|$.

Let

$$\Omega = (\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta),$$  

(3a)

$$\Omega' = (\sin \theta' \sin \varphi', \sin \theta' \cos \varphi', \cos \theta'),$$  

(3b)

$$\hat{n}_L = (\sin \theta_L \sin \varphi_L, \sin \theta_L \cos \varphi_L, \cos \theta_L).$$  

(3c)

Combining Eqs. (1)-(3) we will get formulas for the thee components of vector $\hat{n}_L$, namely,

$$\sin \theta_L \sin \varphi_L = \frac{\sin \theta \sin \varphi - \sin \theta' \sin \varphi'}{\sqrt{2(1 - \Omega \cdot \Omega')}},$$  

(4a)

$$\sin \theta_L \cos \varphi_L = \frac{\sin \theta \cos \varphi - \sin \theta' \cos \varphi'}{\sqrt{2(1 - \Omega \cdot \Omega')}}$$  

(4b)

$$\cos \theta_L = \frac{\cos \theta - \cos \theta'}{\sqrt{2(1 - \Omega \cdot \Omega')}}.$$  

(4c)

The Eq. (4c) directly evaluates $\mu_L = \cos \theta_L$. The expression for $\varphi_L$ can be derived dividing Eq. (4a) by (4b), namely,
\[
\tan \varphi_L = \frac{\sin \theta \sin \varphi - \sin \theta' \sin \varphi'}{\sin \theta \cos \varphi - \sin \theta' \cos \varphi'} = \frac{\sqrt{1 - \mu^2} \sin \varphi - \sqrt{1 - \mu'^2} \sin \varphi'}{\sqrt{1 - \mu^2} \cos \varphi - \sqrt{1 - \mu'^2} \cos \varphi'}.
\]