Chapter I
(Dec-23-2004)

The Radiation Field and the Radiative Transfer Equation

by

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I.1 The Radiation Field

**Photons:** The energy in the radiation field is assumed carried by point mass-less neutral particles called photons. The energy of a photon $E$ (in Joules) is $h\nu$, where $h = 6.626176\times10^{-34}$ J s (Joules second) is Planck’s constant and $\nu$ is photon frequency (in s$^{-1}$). Frequency is related to wavelength $\lambda$ (in meter) as $\nu = c/\lambda$ where $c = 2.99792458\times10^8$ m s$^{-1}$ is speed of light. Photon travels in straight lines between collisions and is regarded as a point particle, with a position described in Cartesian coordinates by the vector $r=(x,y,z)$ and a direction of photon travel by unit vector $\Omega=(\Omega_x,\Omega_y,\Omega_z)$, $\Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1$.

**Problem 1.1.** The frequency of red light is $\nu = 4.3\times10^{14}$ oscillations per second. What is a wavelength $\lambda$ of red light?

**Particle Distribution Function:** Let $f(r,\nu,\Omega,t)$ denote the distribution function such that the number of photons $dn$ at time $t$ in the volume element $dr$ about the point $r$, with frequency in a frequency interval $\nu$ to $\nu + d\nu$, and traveling along a direction $\Omega$ within solid angle $d\Omega$ (see Problem 1.3) is

$$dn = f \, dr \, d\nu \, d\Omega.$$ (1.1)

In the *frequency domain*, the particle distribution function $f(r,\nu,\Omega,t)$ has units of photon number per m$^3$ per frequency interval per steradian (m$^{-3}$ s sr$^{-1}$). In the above definition, one can use the *wavelength* interval $\lambda$ to $\lambda + d\lambda$ instead of its frequency counterpart to define the particle distribution function. In the *wavelength domain*, however, the particle distribution function has units of photon number per m$^3$ per m per steradian (m$^{-4}$ sr$^{-1}$).

**Problem 1.2.** How particle distribution functions in frequency and wavelength domains are related?

It is often convenient to specify the unit vector $\Omega$ in a polar coordinate system, i.e., by the polar angle $\theta$ and the azimuthal angle $\varphi$ as shown in Fig. 1. Cartesian coordinates of $\Omega$ are then

$$\Omega_x = \sin \theta \cos \varphi, \quad \Omega_y = \sin \theta \sin \varphi, \quad \Omega_z = \cos \theta.$$
Figure 1.1. Representation of the unit vector $\Omega = (\Omega_x, \Omega_y, \Omega_z)$, $\Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1$, in a polar coordinate system. Here $\theta$ and $\phi$ are polar and azimuthal angles; $\Omega_x$, $\Omega_y$ and $\Omega_z$ are cartesian coordinates of $\Omega$.

Problem I.3. A solid angle is a part of space bounded by the line segment going from a point (the vertex), to all points of a closed curve. A cone is an example of the solid angle which is bounded by lines from a fixed point to all points on a given circle. The solid angle represents the visual angle under which all points of the given curve can be seen from the vertex. A measure, or “size”, of a solid angle is the area of that part of the unit sphere with center at vertex that is cut off by the solid angle. Units of the solid angle is expressed in the steradian (sr). For a unit sphere whose area is $4\pi$, its solid angle is $4\pi$ sr. In the polar coordinate system, the differential solid angle $d\Omega$ cuts an area consisting of points with polar and azimuthal angles from intervals $[\theta, \theta + d\theta]$ and $[\phi, \phi + d\phi]$. Show that

$$d\Omega = \sin \theta \, d\theta \, d\phi.$$  

(1.2)

Specific Intensity: Many radiometric devices used in remote sensing respond to radiant energy. It is convenient, therefore, to express the particle distribution in terms of energy that photons transfer. Consider a volume element $dr = d\sigma_\Omega \, dz$ with the base $d\sigma_\Omega$ perpendicular to a direction $\Omega$ and the height $dz$. The number of photons in this volume traveling along the direction $\Omega$ is determined by the number of photons which crossed $d\sigma_\Omega$ in the time interval $t$ to $t + dz/c$ where $c$ (in $\text{m s}^{-1}$) is speed of light since the distance traversed by a photon within the interval $dt = dz/c$ does not exceed $dz$. Equation (1.1) can be rewritten as $dn = f \, d\sigma_\Omega \, c \, dt \, dv \, d\Omega$. Since the energy of one photon is $h\nu$, the amount, $dE$, of radiant energy (in J) in a time interval $dt$ and in the frequency interval $\nu$ to $\nu + d\nu$, which crosses a surface element $d\sigma_\Omega$ perpendicular to $\Omega$ within a solid angle $d\Omega$ is given by

$$dE = h\nu \, dn = ch \, f(r,\nu,\Omega,t) \, d\sigma_\Omega \, dt \, dv \, d\Omega.$$  

(1.3)

The distribution of energy that photons transfer is given by specific intensity defined as

$$I(r,\nu,\Omega,t) = c \, h \, \nu \, f(r,\nu,\Omega,t).$$  

(1.4)

Its units is $\text{J m}^{-2} \text{sr}^{-1}$ in the frequency domain and watt $\text{m}^{-3} \text{sr}^{-1}$ in the wavelength domain. If the specific intensity is independent of $\Omega$ at a point, it is said to be isotropic at that point. If the intensity is independent of both $r$ and $\Omega$, the radiation field is said to be homogeneous and isotropic.
Problem 1.4. How the intensities in frequency and wavelength domains are related?

Problem 1.5. Some measuring instruments (e.g., LICOR quantum sensor) register the radiation fluxes in mol m$^{-2}$ s$^{-1}$. Therefore, it is often convenient to use the intensity $J(r,\nu,\Omega,t)$ expressed in mol m$^{-2}$ s$^{-1}$ sr$^{-1}$ instead of $I(r,\nu,\Omega,t)$ in J m$^{-2}$ sr$^{-1}$. How intensities $J$, $I$ and the particle distribution function $f$ are related?

Figure 2 shows an example where a photon beam incident on an area $d\sigma$ at an angle $\theta$ to the normal $n_0$ to $d\sigma$. It is clear that the number of photon crossing the area $d\sigma$ coincides with the number of photons crossing its projected area $d\sigma_\Omega$. Thus, the amount of radiant energy $dE$ (in Joules) in a time interval $dt$, in the frequency interval $\nu$ to $\nu + d\nu$, which crosses a surface element $d\sigma$ in directions confined to a differential solid angle $d\Omega$, which is oriented at an angle $\theta$ to the normal $n_0$ of $d\sigma$ can be expressed as

$$dE = I(r,\nu,\Omega,t) \cos \theta \, d\nu \, d\Omega \, d\sigma \, dt. \tag{1.5}$$

It should be emphasized that the particle distribution function $f$ describes photons at time $t$ while the specific intensity $I$ refers to radiant energy passing a unit area in the time interval $t$ to $t+dt$.

![Figure 1.2. A photon beam incident on the area $d\sigma$ at the angle $\theta$ to the normal $n_0$. Here $d\sigma_\Omega$ is the projection of the area $d\sigma$ onto a plane perpendicular to a direction $\Omega$ of photon travel. Its area can be expressed as $d\sigma_\Omega = d\sigma \cos \theta$. Note that $\cos \theta = |\Omega \cdot n_0|$ where $\Omega \cdot n_0$ is the scalar product of two unit vectors $\Omega$ and $n_0$.](image)

**Energy Density**: Energy density $u$ (Joules/3m$^3$) is the first angular moment of the specific intensity,

$$u(r,t) = \frac{1}{c} \int_0^\infty d\nu \int_{4\pi} d\Omega \, I(r,\nu,\Omega,t). \tag{1.6}$$

**Radiative Flux**: The rate of energy flow per unit area across a surface is defined as the radiative flux. It is a vector quantity. For instance, the $x$ component of the radiative flux $F_x$ (in J m$^{-2}$ s$^{-1}$) is the flow of energy across a unit surface area of an element oriented perpendicular to the $X$ axis, that is,

$$F_x(r,t) = \int_0^\infty d\nu \int_{4\pi} d\Omega \, |\Omega_x| \, I(r,\nu,\Omega,t), \tag{1.7}$$
where $\Omega_x$ is the projection of $\Omega$ along the X axis (Fig. 1). In a similar way, $F_y$ and $F_z$ can be defined.

**Radiation Pressure**: The rate of momentum flow across a surface is defined as pressure. For example, the component $p_{xy}$ (in J m$^{-3}$) is defined as the rate of $y$ momentum flow per unit area through a surface perpendicular to the X axis, and is given by,

$$p_{xy}(r, t) = \frac{1}{c} \int_{0}^{\infty} \int_{4\pi} |\Omega_x, \Omega_y| I(r, v, \Omega, t). \quad (1.8)$$

The momentum of a photon is $h\nu/c$. The other 8 components of the pressure tensor can be similarly introduced.

**I.2. Interaction of Radiation with Matter**

**Absorption**: The absorption coefficient $\sigma_a$ (in m$^{-1}$) is defined such that the probability of a photon being absorbed while traveling a distance $ds$ is $\sigma_a(r, \nu, \Omega, t)ds$. An absorption event signifies true loss of a photon from the count.

**Scattering**: The scattering coefficient $\sigma_s'$ (in m$^{-1}$) is defined in analogy to the absorption coefficient,

$$\text{Probability of scattering} = \sigma_s'(r, \nu, \Omega, t) \, ds.$$

Unlike absorption, a scattering event serves to change the direction and/or frequency of the incident photon. Thus, it is convenient to define a differential scattering coefficient $\sigma_s$ (in m$^{-1}$ sr$^{-1}$) as,

$$\text{Probability of scattering} = \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) \, ds \, d\Omega.$$

The change in photon frequency as a result of a scattering event is not relevant in optical remote sensing of vegetation. It is important to note that photon scattering in vegetation media depends on the absolute coordinates of $\Omega'$ and $\Omega$ in general. The scattering coefficients $\sigma_s'$ and $\sigma_s$ are related as

$$\sigma_s'(r, \nu', \Omega', t) = \int_{0}^{\infty} \int_{4\pi} \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t). \quad (1.9)$$

In some cases, the differential scattering coefficient is decomposed into the product

$$\sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) = \sigma_s'(r, \nu', \Omega', t) K(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t), \quad (1.10)$$
such that, the kernel $K$, termed a scattering phase function, has the interpretation of a probability density function,

$$
\int_0^\infty dv \int d\Omega \ K(r, v' \rightarrow v, \Omega' \rightarrow \Omega, t) = 1.
$$

(1.11)

In the case of coherent scattering, there is no frequency change upon scattering and

$$
K(r, v' \rightarrow v, \Omega' \rightarrow \Omega, t) = K(r, \Omega' \rightarrow \Omega, t) \delta(v' - v)
$$

(1.11a)

where $\delta$ is the Dirac delta function. In the case of isotropic scattering,

$$
K(r, v' \rightarrow v, \Omega' \rightarrow \Omega, t) = \frac{1}{4\pi} K(r, v' \rightarrow v, t).
$$

Therefore, the simplest scattering kernel responds to isotropic coherent scattering, namely,

$$
K(r, v' \rightarrow v, \Omega' \rightarrow \Omega, t) = \frac{1}{4\pi} \delta(v' - v).
$$

(1.12)

**Extinction:** The extinction or the total interaction coefficient $\sigma$ (in m$^{-1}$) is simply the sum $\sigma_a + \sigma_s$. Therefore, $\sigma(r, v, \Omega, t) ds$ is the probability that a photon would disappear from the beam while traveling a distance $ds$ in the medium (note that it can reappear at a different frequency and/or direction.) The quantity $1/\sigma$ denotes photon mean free path, that is, the average distance a photon will travel in the medium before suffering a collision. The dependence on the direction of photon travel is noteworthy and is especially important in the case of vegetation media.

**Single Scattering Albedo:** The probability of scattering given that a collision has occurred is given by the single scattering albedo, $\omega = \sigma_s/\sigma$ (dimensionless). In the case of conservative scattering, $\omega = 1$. The case $\omega = 0$ corresponds to pure absorption.

**Emission:** Photons can be introduced into the medium through external and/or internal sources. In the frequency domain, the number of photons emitted per unit time and volume at frequency $v$ in the interval $v$ to $v + dv$ and in the direction $\Omega$ about the differential solid angle $d\Omega$ is $q(r, v, \Omega, t) dv d\Omega$.

It should be noted that we neglect photon-photon in the above definitions. This means that the photon density is low, that is, low enough such that the overlap in the tails of wavepackets of two photons is negligibly small. This is especially required in the case of source photons emitted at the same location. We also assume that collisions and emission processes occur instantaneously. This imposes a limit on the time resolution over which the above definitions are applicable.
I.3. The Equation of Transfer

The equation of transfer express the law of energy conservation formulated for each direction and for each spatial point in a domain V bounded by a surface $\delta V$. In addition to variables described earlier, one needs to specify the domain V where radiative transfer occurs and its boundary $\delta V$. Their specification is problem dependent. For example, V can be a tree crown, or a part of vegetation canopy with several trees, etc. Here V and $\delta V$ are assumed to be known.

With Eq. (1.1) in mind, we consider a volume element $dr = \Delta S \Delta \xi$ with the base $\Delta S$ perpendicular to a direction $\Omega$ and the height $\Delta \xi$ which is located in a domain V bounded by a surface $\delta V$ (Figure 3.1). The number of photons in this volume element at time $t$ is

$$f(r_1, \nu, \Omega, t) \Delta S \Delta \xi \Delta \nu \Delta \Omega,$$

where $f$ is the photon distribution function. It should be noted that this equation gives the number of photons in the volume element “just before” interaction with matter. In other word, these photons have not yet undergone interaction in $dr$ at time $t$. Number of photons in the volume element at time $t + \Delta t$ before interaction can be specified by counting photons that leave the volume through its lower boundary within the time interval $\Delta t = \Delta \xi / c$ where $c$ is speed of light. This count is given by

$$f(r_2, \nu, \Omega, t) \Delta S \Delta \xi \Delta \nu \Delta \Omega,$$

where $r_2 = r_1 + \Delta \xi \Omega = r_1 + c\Delta t \Omega$. The change in the number of photons in the volume in the time interval $\Delta t$ is

$$\text{change} = \Delta f \Delta S \Delta \xi \Delta \nu \Delta \Omega,$$

where

$$\Delta f = f(r_2, \nu, \Omega, t + \Delta t) - f(r_1, \nu, \Omega, t). \quad (1.13)$$
Problem 1.6. Let $x, y$ and $z$ be Cartesian coordinates of the point $r_1$. Find Cartesian coordinates of the point $r_2 = r_1 + c\Delta t\Omega$.

Problem 1.7. Location $r_1(t)$ of a photon at time $t$ traveling along a direction $\Omega$ can be expressed as $r_1(t) = r_B + c t \Omega$ (Figure 1.3) is the distance traversed by a photon in time interval $t$. Let $x_B, y_B$ and $z_B$ be Cartesian coordinates of the point $r_B$. Find Cartesian coordinates of points $r_1(t)$ and $r_2(t) = r_1 + \Delta \xi \Omega$ and their derivatives with respect to $t$.

In the increment (1.13), the spatial variable $r$ depends on $t$ (see Problems 1.6 and 1.7). Using the chain rule for function of several variables, one gets

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t},$$

where $x, y$ and $z$ are Cartesian coordinates of the point $r_1(t)$ (see Problem 1.7). Thus,

$$\text{change} = \left( \frac{\partial f}{\partial t} + c \Omega_x \frac{\partial f}{\partial x} + c \Omega_y \frac{\partial f}{\partial y} + c \Omega_z \frac{\partial f}{\partial z} \right) \Delta t \Delta s \Delta \xi \Delta \nu \Delta \Omega,$$

where $\Omega_x, \Omega_y$ and $\Omega_z$ are Cartesian coordinates of the unit vector $\Omega$. The first term, $\partial f/\partial t$, in parentheses is the time rate of change of the number of photons. The reminder terms represent a derivative $\Omega \cdot \nabla f$ at $r$ along the direction $\Omega$ which shows the net rate of photons streaming out of the volume element along the direction $\Omega$. Thus,

$$\text{change} = \frac{\partial f}{\partial t} \Delta t \Delta s \Delta \xi \Delta \nu \Delta \Omega + (\Omega \cdot \nabla f) \Delta t \Delta s \Delta \xi \Delta \nu \Delta \Omega \ . \ (1.14)$$

Here $\nabla$ is the vector operator, called “nabla.” Given a scalar function $f$, vector $\nabla f$ has the form

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The change described by Eq. (1.14) is due to four processes – absorption, outscattering, inscattering and emission.

Absorption: A fraction of photons resided in the volume element $dr$ (Fig. 1.3) will be absorbed while traveling a distance $\Delta \xi = c \Delta t$ along the direction $\Omega$. This fraction is determined by the probability $\sigma_a \Delta \xi$ (see Sect. I.2). Thus, the number of absorbed photons is

$$\text{absorption} = f(r, v, \Omega, t) \Delta S \Delta \xi \Delta \nu \Delta \Omega \sigma_a (r, v, \Omega, t) \Delta \xi = c \sigma_a (r, v, \Omega, t) f(r, v, \Omega, t) \Delta S \Delta t \Delta \xi \Delta \nu \Delta \Omega \ . \ (1.15)$$
Outscattering: Another fraction of photons in the volume element \( r \) traveling in the direction \( \Omega \) will change their direction and/or frequency as a result of interaction with matter. The number of photons “lost” due to outscattering from \( \nu \), \( \Omega \) to all other frequencies and directions while traveling a distance \( \Delta \xi = c \Delta t \) is given by

\[
\text{outscattering} = c \Delta t \Delta \xi \Delta \nu \Delta \Omega \int_0^\infty d\nu' \int d\Omega' \, \sigma_s(r, \nu \rightarrow \nu', \Omega \rightarrow \Omega', t) f(r, \nu, \Omega, t),
\]

\[
= c \sigma'(r, \nu, \Omega, t) f(r, \nu, \Omega, t) \Delta t \Delta \xi \Delta \nu \Delta \Omega
\]

(1.16)

Inscattering: Similarly, the number of photons gained due to inscattering to \( \nu \), \( \Omega \) from all other frequencies and directions can be evaluated as

\[
\text{inscattering} = c \Delta t \Delta \xi \Delta \nu \Delta \Omega \int_0^\infty d\nu' \int d\Omega' \, \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) f(r, \nu', \Omega', t).
\]

(1.17)

The rate of the production of photons in the volume element is simply

\[
\text{emission} = c q(r, \nu, \Omega, t) \Delta t \Delta \xi \Delta \nu \Delta \Omega.
\]

(1.18)

Transfer Equation: The equation of transfer is essentially a statement of photon number conservation arrived at by equating the sum of the four terms, Eqs. (1.15) to (1.18), with appropriate signs to designate a loss or gain, to the overall rate of change given by Eq. (1.14):

\[
\text{Change} = - \text{absorption} - \text{outscattering} + \text{inscattering} + \text{emission},
\]

or, after dividing all terms by \( c \Delta t \Delta \xi \Delta \nu \Delta \Omega \) and accounting for the definition of the extinction coefficient \( \sigma = \sigma_a + \sigma' \), one gets

\[
\frac{1}{c} \frac{\partial f}{\partial t} + (\Omega \cdot \nabla f) + \sigma(r, \nu, \Omega, t)f(r, \nu, \Omega, t) = \int_0^\infty d\nu' \int d\Omega' \, \sigma_s(r, \nu' \rightarrow \nu, \Omega' \rightarrow \Omega, t) f(r, \nu', \Omega', t) + q(r, \nu, \Omega, t).
\]

(1.19)

It should be noted that this equation gives the expected or mean value of the photon density distribution. Fluctuations about the mean are not considered. The derived equation also assumes unpolarized light. Four parameters are required to specify the state of polarization of a beam of light, and accordingly, a proper description of photon transport including polarization effects involves four coupled equations of transfer. Assuming the light to be unpolarized by the medium, these equations can be averaged to derive a single equation of transfer and this involves some error. Finally, the transport equation (1.19) does not describe behavior resulting from interference of waves. Therefore, the transfer equation is valid only when the distance between scatterers is large compared to the wave packets.
Sometimes, the second and the third terms on the left hand side of Eq. (1.19) are grouped together; the term $[\Omega \cdot \nabla + \sigma]$ then denotes the **streaming-collision** operator. The transfer equation is clearly an integro-differential equation, which because of its partly differential nature, requires both spatial and temporal boundary conditions.

### I.4. Boundary Conditions

Equation (1.19) describes radiative transfer in a domain $V$ of arbitrary composition and shape bounded by a surface $\delta V$. Boundary conditions in space and time variables are required since the equation of transfer is a first order differential equation in these variables. The spatial boundary condition specifies the radiation field incident at all points on the surface $\delta V$,

$$I(r_n, \nu, \Omega, t) = B(r_n, \nu, \Omega, t), \quad r_n \in \delta V, \quad n(r_n) \cdot \Omega < 0 \quad (1.20)$$

where $B$ is a specified function, $r_n$ is a point on the surface $\delta V$ and $n(r_n)$ is an outward normal vector at this point. The case of **vacuum** boundary condition refers to $B = 0$. The temporal boundary condition is

$$I(r, \nu, \Omega, 0) = \Lambda(r, \nu, \Omega), \quad r \in V, \quad (1.21)$$

on the assumption that the temporal range of interest is $0 < t < \infty$. The radiative transfer problem is thus completely specified by the equation of transfer [Eq. (1.19)] and the two boundary conditions [Eqs. (1.20) and (1.21)].

### I.5. Steady-State Radiative Transfer Problem

If the extinction and differential scattering coefficients, emission and the boundary condition do not change with time, $\partial I/\partial t = 0$ and the radiative transfer problem becomes a steady-state radiative transfer problem. In the case of coherent scattering, the boundary value problem for radiative transfer equation in the **wavelength domain** rearranges to the form

$$\Omega \cdot \nabla I_\lambda + \sigma_\lambda(r, \nu, \Omega)I_\lambda(r, \Omega) = \int d\Omega' \sigma_{\lambda,\lambda}(r, \Omega' \rightarrow \Omega) I_\lambda(r, \Omega') + q_\lambda(r, \Omega), \quad (1.22)$$

$$I_\lambda(r_B, \Omega) = B_\lambda(r_B, \Omega), \quad r_B \in \delta V, \quad n(r_B) \cdot \Omega < 0. \quad (1.23)$$

Here $I_\lambda(r, \Omega)$ is the **monochromatic** specific intensity which depend on wavelength $\lambda$, location $r$ and direction $\Omega$. The emission term $q_\lambda$ is required at thermal wavelengths, but $q_\lambda = 0$ at solar wavelengths. Note that the wavelength $\lambda$ is a parameter of the radiative transfer problem. We have emphasized this feature in notations by moving the wavelength from the argument list to subscript.

### I.6. The Law of Energy Conservation

The steady-state radiative transfer problem (1.22)-(1.23) expresses the law of energy conservation for a given domain $V$ in the most general form. The first term in the transport
equation (1.22) characterizes the change in radiance in the direction $\Omega$ at the spatial point $r$ in $V$; the other terms show whether the changes take place at the expense of absorption and outscattering at $r$ (second term), at the expense of the inscattering from other directions (third term). In other words, the transport equation is a statement of energy conservation for each spatial point $r$ within $V$ and for each direction $\Omega$. The boundary condition (1.23) describes energy exchange between $V$ and the surrounding medium. The aim of this section is to derive an expression of the energy conservation law for the domain $V$ bounded by a surface $\delta V$. The domain $V$ is variable in this section. It can be, e.g., a parallelepiped with plants within it, or a crown of an individual tree within the forest.

The following notations are needed to perform integration of the transport equation over the domain $V$ and unit directions $4\pi$. Let $X$ be the phase space of spatial points and directions; that is,

$$X=\{ (r, \Omega): r = (x, y, z) \in V; \Omega \in 4\pi \}. \quad (1.24)$$

Measures of elementary volume $dr$ and solid angle $d\Omega$ induce a corresponding measure in the five dimensional space $X$ as $dx=drd\Omega$. Throughout this section, we will use the notations $r_B$ and $n(r_B)$ to denote a point on the boundary $\delta V$ and the outward normal to $\delta V$ at the point $r_B$. Let us fix a certain direction $\Omega$. By $\pi(r_0, \Omega)$ we mean a plane perpendicular to the direction $\Omega$ and passing a fixed point $r_0$ (Figure 1.4). Let $Q$ be a variable point on the plane $\pi(r_0, \Omega)$. We construct through $Q$ a straight line $\Pi(Q, \Omega)$ parallel to the direction $\Omega$, i.e.,

$$\Pi(Q, \Omega) = \{ Q + \xi\Omega, -\infty < \xi < +\infty \}. \quad (1.25)$$

The phase point $x=(r,\Omega)$ can be uniquely represented as (see Figure 1.4)

$$x = (Q, \xi, \Omega), \quad r = Q + \xi\Omega. \quad (1.26)$$

The elementary volume about the phase point $x=(r,\Omega)$ in $X$ can be expressed as [Vladimirov, 1963]

$$dx = drd\Omega = dQd\xi d\Omega. \quad (1.27)$$

Note that $dQ$ is an elementary surface at the point $Q$ perpendicular to the direction $\Omega$. If we choose another elementary surface $dQ'$ at the point $Q$ perpendicular to another direction $\Omega'$, the expression (1.27) takes the form

$$dx = drd\Omega = |\Omega\cdot\Omega'| d\Omega dQ'd\xi. \quad (1.28)$$
Figure 1.4. Representation of the phase point $x=(r,\Omega)$ within a given volume $V$ of space bounded by a surface $\delta V$. Here $\pi(r_0,\Omega)$ is a plane perpendicular to the direction $\Omega$ and passing a fixed point $r_0$; $\Pi(Q,\Omega)$ is a straight line parallel to the direction $\Omega$ and passing the point $Q$ on the plane $\pi(r_0,\Omega)$; $n(r_B)$ is the outward normal to $\delta V$ at the point $r_B$; $\eta_1$ is the distance between $r_B$ and a point obtained from the intersection of the line $\Pi(Q,\Omega)$ with the boundary $\delta V$. A phase point $(r,\Omega)$ with $r$ on the line $\Pi(Q,\Omega)$ can be represented as $r = Q + \xi \Omega$.

It follows from (1.28) that the following representation is valid

$$
\int \varphi(r,\Omega) \, dr \, d\Omega = \int \int \Omega \cdot \Omega \int_{\Omega(r_0,\Omega)}^{\xi_1} \varphi(Q + \xi \Omega,\Omega) \, d\xi \\
= \int \int_{\delta V} \int_{n(r_B) \cdot \Omega = 0}^{\eta_1} n(r_B) \cdot \Omega \, |d\Omega| \int_{\partial V} \varphi(r_B + \eta \Omega,\Omega) \, d\eta .
$$

Here $\xi_0$ and $\xi_1$ are distances between the point $Q$ and the boundary $\delta V$ and $\eta_1$ is the distance between $r_B$ and a point obtained from the intersection of the line $r_B + \xi \Omega$, $\xi > 0$ with the boundary $\delta V$.

Integrating equation over $X = 4\pi V$ and accounting for (1.9) yields

$$
\int_{4\pi V} \Omega \cdot \nabla \Omega \, d\Omega + \int_{4\pi V} \left[ \sigma(r,\Omega) - \sigma'(r,\Omega) \right] I(r,\Omega) \, d\Omega \, dr = 0 .
$$

Note that we temporarily suppress the wavelength in notations. The difference between $\sigma$ and $\sigma'_s$ is the absorption coefficient (see section I.2). It means that the second term in (1.30) is the
amount of radiant energy in \( d\lambda \) centered at \( \lambda \) absorbed by the domain \( V \) (in W m\(^{-1}\)). We use the symbol \( E_a(V) \) to denote this variable.

Consider the first term. The following transformations are based on the equation (1.29). We have

\[
\int \int \int \int \int \int -\nabla \cdot \Omega = \int \Omega \cdot \nabla I \, d\Omega \, dr = \int \Omega \cdot \nabla I \, d\Omega \, \frac{dI(Q + \xi \Omega, \Omega)}{d\xi} = \int \Omega \frac{dI(Q + \xi \Omega, \Omega) - I(Q - \xi \Omega, \Omega)}{d\xi} = \int \Omega I(Q + \xi \Omega, \Omega) - \int \Omega I(Q - \xi \Omega, \Omega)
\]

\[
= \int \Omega \int d\Omega |\Omega| \Omega I(r_B, \Omega) - \int \Omega \int d\Omega |\Omega| \Omega I(r_B, \Omega)
\]

\[
= \int \int dr \, n(r_B) \cdot \Omega |\Omega| \Omega I(r_B, \Omega) - \int \int dr \, n(r_B) \cdot \Omega |\Omega| \Omega I(r_B, \Omega)
\]

\[
= \int F^+(r_B) \, dr_B - \int F^-(r_B) \, dr_B = E^+ (\delta V) - E^- (\delta V). \quad (1.31)
\]

It follows from (1.30) and (1.31) that the low of energy conservation for a given volume \( V \) bounded by a surface \( \delta V \) can be expressed as

\[
E^+ (\delta V) + E_a(V) = E^- (\delta V), \quad (1.32)
\]

that is, the amount of radiant energy reflected and absorbed by a volume \( V \) is equal to the amount of energy incident on the boundary \( \delta V \) of the volume \( V \).

**Problem 1.8:** Let \( \delta V \) be a reflecting boundary, i.e., a fraction of the medium leaving radiation can be reflected back into \( V \). Assume that boundary reflect as a Lambertian surface. The radiation \( I^+ \) penetrating into \( V \) through \( \delta V \) is

\[
I(r_B, \Omega) = \frac{p}{\pi} \int I(r_B, \Omega') |n(r_B) \cdot \Omega| \, d\Omega' + q(r_B, \Omega), \quad n(r_B) \cdot \Omega < 0.
\]

Show that \((1-p)E^+ (\delta V) + E_a(V) = E^- (\delta V)\), where

\[
E^- (\delta V) = \int \int dr \, n(r_B) \cdot \Omega |q(r_B, \Omega)|.
\]

**Problem 1.9:** Let \( V \) be the parallelepiped and \( \delta V_t, \delta V_b, \delta V_b \) are its top, bottom and lateral surfaces. Show that

\[
E^+ (\delta V) = E^+ (\delta V_b) + E^+ (\delta V_t) + E^+ (\delta V_l); \quad E^- (\delta V) = E^- (\delta V_b) + E^- (\delta V_t) + E^- (\delta V_l).
\]

**Problem 1.10:** Let \( V \) be the parallelepiped and \( \delta V_t, \delta V_b, \delta V_b \) are its top, bottom and lateral surfaces. Write the energy conservation low in terms of canopy transmission, \( t \), reflection, \( r \), and horizontal energy flow, \( h \), defined as
\[ r = \int_{n(r) \Omega \leq 0} I(r, \Omega') |n(r') \Omega'| d\Omega', \quad t = \int_{n(r) \Omega \leq 0} I(r, \Omega') |n(r') \Omega'| d\Omega', \quad h = \int_{n(r) \Omega \leq 0} I(r, \Omega') |n(r') \Omega'| d\Omega'. \]

### I.7. The Equation of Transfer in Integral Form

It is instructive to derive an integral form of the equation of transfer for the physical insight it provides into the process of radiation transport. Consider the boundary value problem for steady-state radiative transfer equation (1.22)-(1.23). We rewrite Eq. (1.22) as

\[ \Omega \cdot \nabla I(r, \Omega) + \sigma(r, \Omega) I(r, \Omega) = Q(r, \Omega), \quad (1.33) \]

where \( Q \) denotes the angular source due to inscattering and emission. In this section we suppress the wavelength dependence in notations. This equation can be written as (see Fig. 1.4)

\[ \frac{dI(r_B + \xi \Omega, \Omega)}{d\xi} + \sigma(r_B + \xi \Omega, \Omega) I(r_B + \xi \Omega, \Omega) = Q(r_B + \xi \Omega, \Omega). \quad (1.34) \]

Multiplying the above with exponential integrating factors,

\[ \exp \left( \int_0^\xi d\xi'' \sigma(r_B + \xi'' \Omega, \Omega) \right) \]

yields,

\[ \frac{d}{d\xi} \left\{ I(r_B + \xi \Omega, \Omega) \exp \left[ \int_0^\xi d\xi'' \sigma(r_B + \xi'' \Omega, \Omega) \right] \right\} = Q(r_B + \xi \Omega, \Omega) \exp \left[ \int_0^\xi d\xi'' \sigma(r_B + \xi'' \Omega, \Omega) \right]. \]

Integration of this equation from 0 to \( \xi \) results in

\[ I(r_B + \xi \Omega, \Omega) = I(r_B, \Omega) \exp \left( -\int_0^\xi d\xi' \sigma(r_B + \xi' \Omega, \Omega) \right) + \int_0^\xi d\xi' Q(r_B + \xi' \Omega, \Omega) \exp \left( -\int_0^\xi d\xi'' \sigma(r_B + \xi'' \Omega, \Omega) \right). \quad (1.35) \]

Note that since \( n(r_B) \Omega < 0 \), \( I(r_B, \Omega) \) coincides with \( \Gamma(r_B, \Omega) \) which is a specified function. Substituting

\[ Q(r, \Omega) = \int_{4\pi} \sigma_{\omega} (r, \Omega' \rightarrow \Omega) I_{\lambda'} (r, \Omega') + q_{\lambda} (r, \Omega) \]

into (1.35), one gets the desired integral equation for the specific intensity \( I \). It can be treated as a solution to the radiative transfer equation (1.33) given \( Q \) and the boundary condition (1.23). The first term in (1.35) describes the attenuation of uncollided intensity \( I \), i.e., photons that arrive at \( r \)
along direction $\Omega$ without experiencing a collision. Also this term describes radiation field in purely absorbing medium ($Q=0$).

The physical interpretation of Eq. (1.35) is as follows (Fig. I.4). The specific intensity $I$ at a point $r$ in the direction $\Omega$ is the sum of two field − uncollided and collided radiation. The uncollided component is the intensity of photons entering $V$ through the point $r_B$ and exponentially attenuated by collisions while traveling the distance between points $r_B$ and $r$. The collided part at the point $r$ accounts for photons inscattered at points $r'$ between $r_B$ and $r$ and attenuated while traveling the distance between $r'$ and $r$. The quantity $\tau(r_1, r_2, \Omega)$ defined as

$$\tau(r_1, r_2, \Omega) = \int_{0}^{||r_1 - r_2||} d\xi \sigma(r_1 + \xi \Omega)$$

(1.36)

is referred to as the optical distance between the points $r_1$ and $r_2$. Here $||r_1 - r_2||$ is the geometrical distance between the points $r_1$ and $r_2$. The estimation of $\tau$ from measurements of $I$ is a principle problem in remote sensing.

**Problem 1.11.** Show that if the extinction coefficient $\sigma$ does not depend neither on spatial nor angular variables, $\tau(r_1, r_2, \Omega) = \sigma ||r_1 - r_2||$.

**Problem 1.12.** Express the integral equation (1.35) in terms of optical depth.

### I.8. Uniqueness Theorems

Here we formulate conditions under which the boundary value problem for the steady-state radiative transfer problem has a unique solution. Consider a domain $V$ bounded by a reflecting surface $\delta V$. The boundary condition (1.23) is given by the following equation

$$B_\lambda(r_B, \Omega) = \frac{1}{\pi n(\rho_0, \lambda)} \int_{\Omega} \rho_\lambda(r_B, \Omega' \rightarrow \Omega) d\Omega' + q_\lambda(r_B, \Omega)$$

(1.37)

Here $\rho_\lambda(r_B, \Omega' \rightarrow \Omega)$ is the bidirectional reflectance factor of the surface $\delta V$; that is, the probability density that a photon impinging on a boundary surface element at $r_B$ along direction $\Omega'$ will be reflected back to the domain $V$ in the direction $\Omega$; and $q_\lambda(r_B, \Omega)$ is a photon source at the canopy boundary $\delta V$. The bidirectional reflectance factor is said to be symmetric if $\rho_\lambda(r_B, \Omega' \rightarrow \Omega) = \rho_\lambda(r_B, \Omega \rightarrow \Omega')$.

The following parameters characterize optical properties of scatters and the entire medium as well as medium-boundary interaction.

**The maximum boundary albedo**, $\rho_{0,\lambda}$, quantifies the magnitude of boundary scattering which is defined as
$$\rho_{0,\lambda} = \sup_{r_B \in \partial \Omega} \frac{1}{\Omega} \int \rho_{\lambda}(r_B, \Omega' \rightarrow \Omega) |n(r_B) \cdot \Omega| \, d\Omega.$$ (1.38)

**The maximum optical depth:** The optical distance between two points in the domain \(V\) is given by Eq. (1.36). We denote its maximum value by \(\tau_{0,\lambda}(V)\), i.e.,

$$\tau_{0,\lambda}(V) = \sup_{r, r' \in V, \Omega \in 4\pi} \tau_{\lambda}(r, r', \Omega).$$ (1.39)

**The maximum single scattering albedo.** The single scattering albedo is the probability that photon will be scattered as a result of collision (Section I.2). Its maximum value \(\sigma_{\lambda}(V)\) is

$$\sigma_{\lambda}(V) = \sup_{r \in V, \Omega \in 4\pi} \sigma(r, \Omega).$$ (1.40)

These variables depend on wavelength \(\lambda\), which is a parameter of the radiative transfer problem. We will omit the sign \(\lambda\) denoting the wavelength dependence in further notations.

The following theorem is a special case of Germogenova’s maximum principle which is proved here under assumption of a symmetry property for the differential scattering coefficient, namely, \(\sigma_{\lambda}(r, \Omega' \rightarrow \Omega) = \sigma_{\lambda}(r, \Omega \rightarrow \Omega')\). We will show in next Section that this restriction can be relaxed.

**Theorem 1.** Let \(I(r, \Omega)\) satisfies Eq. (1.22) in the domain \(V\) and \(\sigma(V) \leq 1\), \(\tau_{0}(V) < \infty\) and \(q_{\lambda} = 0\). The following inequality holds true

$$|I(r, \Omega)| \leq \sup_{r_B \in \partial \Omega, \Omega \in 4\pi} |I(r_B, \Omega)|,$$ (1.41)

for all \(r \in V + \delta V\) and \(\Omega \in 4\pi\). Here \(n(r_B)\) is the outward normal to the boundary \(\delta V\) at the point \(r_B \in \delta V\).

This theorem states that the intensity of radiation within the vegetation canopy can not exceed a maximum value of the intensity of radiation penetrating into the canopy through the boundary \(\delta V\). This theorem also presupposes that the incident radiation field is described by a bounded function. It means that this theorem can not be applied if the incident radiation is given by a delta function.

**Proof.** Let \(\overline{I} = \sup_{r \in V, \Omega \in 4\pi} |I(r, \Omega)|\) where “supremum” is taken over all spatial points in \(V\) and over all directions. We have

$$\Omega \cdot \nabla I(r, \Omega) = -\sigma(r, \Omega)I(r, \Omega) + \int d\Omega' \sigma_{\lambda}(r, \Omega' \rightarrow \Omega)I(r, \Omega')$$

$$\leq -\sigma(r, \Omega)I(r, \Omega) + \int d\Omega' \sigma_{\lambda}(r, \Omega' \rightarrow \Omega) \sup_{r \in V, \Omega \in 4\pi} \{I(r, \Omega')\}$$

this term is constant and equal to \(\overline{I}\).
\[ \leq -\sigma_s(r,\Omega)I_s(r,\Omega) + \frac{\Omega \cdot \nabla[I-I(r,\Omega)\sigma(r,\Omega)]}{|I-I(r,\Omega)|} \leq |I-I(r,\Omega)| \sigma(r,\Omega). \]  

(1.42)

Comparing the first and last term in (1.42), one can obtain

\[ [I-I(r,\Omega)]\sigma(r,\Omega) + \frac{\Omega \cdot \nabla[I-I(r,\Omega)\sigma(r,\Omega)]}{|I-I(r,\Omega)|} \geq 0. \]

(1.43)

Multiplying the above by \(\exp(-\tau(r,r-\xi\Omega,\Omega))\), yields

\[-\frac{d}{d\xi} [I-I(r-\xi\Omega,\Omega)] \exp(-\tau(r,r-\xi\Omega,\Omega)) \geq 0.\]

Integrating the above over the interval \([0,\xi]\) results in

\[ [I-I(r-\xi\Omega,\Omega)] \exp(-\tau(r,r-\xi\Omega,\Omega)) \leq [I-I(r,\Omega)] . \]

(1.44)

Let us assume that the solution \(I(r,\Omega)\) reaches its maximum at a point \(r_0\) within \(V\) and in direction \(\Omega_0\), i.e., \(I=I(r_0,\Omega_0)\). Let \(\xi_B\) be the distance between the point \(r_0\) and the boundary \(\delta V\) along the direction \(-\Omega_0\) opposite to \(\Omega_0\). It follows from (1.44) that

\[ 0 \leq [I(r_0,\Omega_0) - I(r_0 - \xi_B\Omega_0,\Omega_0)] \exp(-\tau(r_0,r_0 - \xi_B\Omega_0,\Omega_0)) \leq I(r_0,\Omega_0) - I(r_0,\Omega_0) = 0, \]

which holds true if and only if \(I(r_0,\Omega_0) = I(r_0 - \xi_B\Omega_0,\Omega_0)\). It means that the maximum of the solution \(I(r,\Omega)\) taken over all internal point and over all directions can not exceed intensity of radiation entering the canopy in the direction \(\Omega_0\) through the point \(r_0\) on the boundary \(\delta V\). This completes the proof.

Inequality (1.44) for a more general case was originally derived by Germogenova [1986]. This results provides theoretical justification to many existing radiation models. Based on Theorem 1, the following uniqueness theorem can be easily proved under assumption of the symmetry property for the differential scattering coefficient \(\sigma_s\) and the boundary bidirectional reflectance factor \(\rho\).

**Uniqueness Theorem.** Let \(\sigma \leq 1, \rho_0 < 1\) and \(\tau_0(V) < \infty\). The radiative regime within a given volume \(V\) of space bounded by the surface \(\delta V\) is uniquely determined by the boundary conditions (1.37).

**Proof.** Let \(I_1(r,\Omega)\) and \(I_2(r,\Omega)\) be two solutions of the transport equation subject to given conditions. The function \(\psi(r,\Omega) = I_1(r,\Omega) - I_2(r,\Omega)\) satisfies Eq. (1.22) and the boundary
condition (1.37) with $\psi_\lambda = 0$. It follows from Theorem 1 and the symmetry property $\rho(r_B, \Omega \rightarrow \Omega') = \rho(r_B, \Omega' \rightarrow \Omega)$ that the inequality

$$|\psi(r, \Omega)| \leq \sup_{r_B \in \delta\Omega'} |B(r_B, \Omega)| = \sup_{r_B \in \delta\Omega'} \left| \int \rho(r_B, \Omega' \rightarrow \Omega) |n(r_B) \cdot \Omega'| \psi(r_B, \Omega') \, d\Omega' \right|$$

$$\leq \rho_0 \sup_{r_B \in \delta\Omega'} |\psi(r_B, \Omega)|,$$

where $\rho_0 < 1$, the latter holds true if and only if $|\psi(r, \Omega)| = 0$, i.e., $I_1(r, \Omega) = I_2(r, \Omega)$. The uniqueness theorem is proved.

**I.9. Release the symmetry conditions**

Theorem 1 and consequently the uniqueness theorem are proved under the assumption that $\sigma_s(r, \Omega' \rightarrow \Omega) = \sigma_s(r, \Omega \rightarrow \Omega)$ (symmetry property for the differential scattering coefficient) and $\rho(r_B, \Omega \rightarrow \Omega') = \rho(r_B, \Omega' \rightarrow \Omega)$ (symmetry property for the boundary bidirectional reflectance factor). This assumption was required to obtain $\sigma$ in Eq. (1.42) and $\rho_0$ in Eq. (1.45). To extend the validity of the uniqueness theorem to the general case, consider an adjoint formulation of the transport equation [Bell and Glasstone, 1970; Germogenova, 1986], i.e.,

$$-\Omega \cdot \nabla I^*(r, \Omega) + \sigma(r, \Omega) I^*(r, \Omega) = \int_{4\pi} \sigma_s(r, \Omega \rightarrow \Omega') I^*(r, \Omega') \, d\Omega',$$

$$I^*(r_B, \Omega) = B^*(r_B, \Omega), \quad r_B \in \delta\Omega', \quad n(r_B) \cdot \Omega > 0,$$

where

$$B^*(r_B, \Omega) = \frac{1}{\pi} \int \rho(r_B, \Omega \rightarrow \Omega') |n(r_B) \cdot \Omega'| I^*(r_B, \Omega') \, d\Omega' + q^*(r_B, \Omega), \quad n(r_B) \cdot \Omega > 0.$$
Physically, the adjoint radiative transfer problem describes the time-reversed photon flow. This gives us the hint that adjoint sources $q^*$ describe the position of detectors while the adjoint transport equation takes them backwards in time to actual sources. Adjoint equations and their solutions play an important role in radiative transfer theory. Adjoint functions are, in a very real sense, orthogonal to the solutions of the radiative transfer equation [Bell and Glasstone, 1970; Germogenova, 1986]. For this and other reasons, they are widely used in perturbation theory and variational calculations relating to the behavior of 3D optical media. The properties of the solutions of the adjoint RTE are also used in the development of effective Monte Carlo calculations [Marchuk et al., 1980].

Consider the function $I_0^\star(r,\Omega) = I^\star(r, -\Omega)$. It satisfies the standard boundary value problem for the standard transport equation, i.e.,

\[
\Omega \cdot \nabla I_0^\star(r,\Omega) + \sigma(r, -\Omega) I_0^\star(r,\Omega) = \int_{4\pi} \sigma_s(r, -\Omega \rightarrow -\Omega') I_0^\star(r, -\Omega') d\Omega' .
\]

(1.49)

\[
I_0^\star(r_B,\Omega) = B^\star(r_B, -\Omega) , ~ r_B \in \mathcal{V}' , ~ n(r_B) \cdot \Omega < 0 .
\]

(1.50)

The uniqueness theorem can be applied to Eqs. (1.49)-(1.50) with the maximum boundary albedo, single scattering albedo and optical depth calculated using $\rho(r_B, -\Omega \rightarrow -\Omega')$, $\sigma(r, -\Omega)$ and $\sigma_s(r, -\Omega \rightarrow -\Omega')$. According to the Fredholm alternative [Bronshtein and Semendyaev, 1985, p. 783], a linear operator equation and its adjoint counterpart have a unique solution simultaneously. Therefore, we can use the adjoint transport equation to find conditions under which it has a unique solution. The same conditions guarantee the uniqueness of the transport equation. Thus, the symmetry for the differential scattering coefficient and the boundary bidirectional reflectance factor can be relaxed [Knyazikhin, 1990].

**Problem I.13:** Prove the uniqueness theorem without assuming the symmetry property for the differential scattering coefficient and the boundary bidirectional reflectance factor.

**Problem I.14:** Formulate the uniqueness theorem for the transport equation with the emission sources $q(r,\Omega)$ within $\mathcal{V}$,

\[
\Omega \cdot \nabla I(r,\Omega) + \sigma(r,\Omega) I(r,\Omega) = \frac{u_L(r)}{\pi} \int_{4\pi} \sigma_s(r,\Omega' \rightarrow \Omega) I(r,\Omega') d\Omega' + q(r,\Omega) ,
\]

and boundary conditions given by Eqs. (1.23) and (1.37).
I.10. Bibliography

Further readings


References