Chapter III  
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Radiative Transfer in Vegetation Canopies

by

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III.1. The Radiative Transfer Equation for Vegetation Canopies

Solar radiation scattered from a vegetation canopy and measured by satellite sensors results from interaction of photons traversing through the foliage medium, bounded at the bottom by a radiatively participating surface. Therefore to estimate the canopy radiation regime, three important features must be carefully formulated. They are (1) the architecture of individual plant and the entire canopy; (2) optical properties of vegetation elements (leaves, stems) and soil; the former depends on physiological conditions (water status, pigment concentration); and (3) atmospheric conditions which determine the incident radiation field [Ross, 1981]. Photon transport theory aims at deriving the solar radiation regime, both within the vegetation canopy and the radiant exitance, using the above mentioned attributes as input data. The leaf area density distribution, $u_L$, leaf normal orientation distribution, $g_L$, leaf scattering phase function, $\gamma_L$, and boundary conditions specify these input.

Let the domain $V$ in which a vegetation canopy is located, be a parallelepiped of horizontal dimensions $X_S$, $Y_S$, and height $Z_S$. The top $\delta V_t$, bottom $\delta V_b$, and lateral $\delta V_l$ surfaces of the parallelepiped form the canopy boundary $\delta V=\delta V_t+\delta V_b+\delta V_l$. Note the boundary $\delta V$ is excluded from the definition of $V$. The function characterizing the radiative field in $V$ (i.e., within the canopy space) is the specific intensity introduced by Eq. (1.4). Under condition of the absence of polarization, frequency shifting interaction, and emission processes within the canopy, the monochromatic specific intensity $I_\lambda(r,\Omega)$ is given by the steady-state radiative transfer equation (1.22) with $q_\lambda(r,\Omega)=0$. Substituting (2.13) and (2.17) into the transport equation (1.22), we obtain the radiative transfer equation for vegetation canopies, namely,

$$\Omega \cdot V I_\lambda(r,\Omega) + G(r,\Omega)u_L(r)I_\lambda(r,\Omega) = \frac{u_L(r)}{\pi} \int G(r,\Omega') \rightarrow \Omega I_\lambda(r,\Omega')d\Omega'. \quad (3.1)$$

The boundary condition for the radiative transfer problem in the vegetation canopy is given by

$$I_\lambda(r_B,\Omega) = B_\lambda(r_B,\Omega), \quad r_B \in \delta V, \quad n(r_B) \cdot \Omega < 0. \quad (3.2)$$

Here $B_\lambda$ is a specified function, $r_B$ is a point on the surface $\delta V$ and $n(r_B)$ is an outward normal vector at $r_B$.

The solution to this equation, i.e., the monochromatic specific intensity $I_\lambda(r,\Omega)$, depends on wavelength, $\lambda$, location $r$, and direction $\Omega$. Here, the position vector $r$ denotes the triplet $(x,y,z)$ with $(0<x<X_S)$, $(0<y<Y_S)$ and $(0<z<Z_S)$ and is expressed in Cartesian coordinates with its origin, $O=(0,0,0)$, at the top of the vegetation canopy and the $Z$ axis directed down into the vegetation canopy. The unit vector $\Omega-\mu, \varphi)$ has an azimuthal angle $\varphi$ measured in the (XY) plane from the positive X axis in a counterclockwise fashion and a polar angle $\theta=\cos(\mu)$ with respect to the polar axis that is opposite to the $Z$ axis. $\Omega \cdot V I_\lambda(r,\Omega)$ is a derivative at $r$ along the direction $\Omega$ that is defined in section I.3. We shall omit the sign $\lambda$ denoting the wavelength dependence in notations.
Juhan Ross, the founder of the theory of radiative transfer in vegetation canopies, wrote [Ross, 1981, p. 144] that in deriving the radiative transfer equation we proceeded from a few contradictory assumptions. On the one hand the assumed elementary volume is so small that no mutual shading exists in any direction within it. On the other hand the number of plates (scatters) in the elementary volume was assumed to be so great that the functions \( u_L \) and \( (1/2\pi)g_L \) are realized to the necessary degree of accuracy. The radiation intensities for which the transfer equation is set down are essentially mean values for the elementary volume. The use of the transport equation, therefore, requires an accurate specification of the elementary volume which depends on the problem one investigates. By other words, the transport equation must be adjusted for a certain application. It can be performed as follows.

Consider a vegetation canopy shown in Figure 2.1. Let the domain V in which the sample site (a square in Figure 2.1) is located be a parallelepiped limited by the slope, a plane parallel to the slope at the height \( Z_S \) of the tallest tree and a lateral surface of horizontal dimensions \( X_S \) and \( Y_S \). We impose a fine spatial cell mesh of the resolution \( \varepsilon \times \varepsilon \times \varepsilon \) on the domain V (Figure 2.2, the left panel). Let us segregate a cell (volume) \( \Delta V(r) \) around a point \( r \) within the stand so that this cell contains phytoelements without considerable mutual shading in any direction. Evaluate the area \( \Delta S(r) \) of leaves within \( \Delta V(r) \). Specify mean leaf scattering phase function \( \gamma_L(\Delta V) \) for \( \Delta V(r) \). Divide the leaf surface within \( \Delta V(r) \) into small elementary sub-cells and define the law of distribution of their normals by \( g_L(\Delta V, r) \). Displacing the volume \( \Delta V(r) \) in any direction within the stand we obtain the values of \( u_L(r) = \Delta S(r)/\Delta V(r) \) as well as the functions \( g_L(\Delta V, r) \) and \( \gamma_L(\Delta V) \). Find a solution, \( I(r, \Omega) \), to the transport equation (3.1.) using these variables. The predicted intensity, \( I(r, \Omega) \), is a function of the cell size \( \Delta V \).

Suppose we have an accurate measurement of the intensity at each fine cell (Figure 2.2, the left panel) and in any direction, i.e., we have \( \bar{I}(r, \Omega) \) derived from field measurements (which does not depends on \( \Delta V \)). Predicted radiation in one cell can greatly differ from the measured one, e.g., due to mistakes in input for the transport equation. Because the transport equation assumes the energy balance for any elementary volume, some other cells in the neighborhood may exist to compensate for this difference. So the mean predicted intensities over cells from this neighborhood will, much better, agree with the mean measured intensities over the same neighborhood. By other word, the mean predicted and measured values over coarser cells will agree better. These coarser cells specify a resolution at which the radiative transfer equation predicts the radiation field and accuracy of the predictions. Thus, we use two attributes to describe the accuracy with which the radiative transfer equation predicts the radiative regime. They are the fine and coarse cells. The first one is used to set down the transfer equation, i.e., to specify \( u_L \), \( g_L \) and \( \gamma_L \) at the fine cell and to find a solution, \( I(r, \Omega) \), to the transport equation at this resolution. The coarse cell specifies a domain over which the solution \( I(r, \Omega) \) must be averaged to predict mean values for the coarse cells with given degree of accuracy. The accuracy depends on the number of fine cells in the coarse resolution. Such an analysis must be followed by any application of the radiative transfer equation to a particular problem [Knyazikhin et al., 1997].
III. 2. Canopy Reflection

The hemispherical-directional reflectance factor (HDRF) for nonisotropic incident radiation is the ratio of the mean radiance leaving the top of the plant canopy to radiance reflected from an ideal Lambertian target into the same beam geometry and illuminated under identical atmospheric conditions [Martonchik et al., 2000]. This variable can be expressed in terms of the solution of the radiative transfer equation as

$$R = \frac{\int_{0}^{\infty} \int_{0}^{\infty} I(x, y, 0, \Omega) \, dx \, dy}{\frac{1}{\pi} \int_{-\pi}^{\pi} |\mu'| \, d\Omega'} \frac{\int_{0}^{\infty} \int_{0}^{\infty} I(x, y, 0, \Omega') \, dx \, dy}{\frac{1}{\pi} \int_{-\pi}^{\pi} \langle I(\Omega) \rangle_0 \, d\Omega'}, \quad \mu > 0. \quad (3.3)$$

Here $\mu$ and $\mu'$ are the cosine of the polar angles of the upward $\Omega$ downward $\Omega'$, respectively; the angle brackets $<>_0$ denotes the mean over the upper surface $\delta V_t$ of the parallelepiped $V$. The HDRF depends on the angular distribution of incoming radiation, the area of the upper boundary $\delta V_t$, the height $Z_S$ and the direction $\Omega$. In remote sensing, the dimension of the upper boundary $\delta V_t$ is called a resolution; the upward direction $\Omega$ is the view direction. For the condition of no atmosphere, i.e., the incident solar radiation at the upper canopy boundary $\delta V_t$ is a parallel beam of light, the HDRF is termed a bidirectional reflectance factor (BRF). We use the symbol $R_0$ to denote its value. The bidirectional reflectance distribution function (BRDF) is a factor of $\pi$ smaller than BRF, i.e., $\pi^{-1}R_0$.

The bihemispherical reflectance (BHR) for nonisotropic incident radiation is the ratio of the mean radiant exitance to the incident radiant [Martonchik et al., 1998], i.e.,

$$A = \frac{\int_{-\pi}^{\pi} \langle I(\Omega) \rangle_0 \, d\Omega}{\frac{1}{\pi} \int_{-\pi}^{\pi} |\mu| \, d\Omega} \cdot (3.4)$$

The BHR does not depend on the view direction. However, it depends on the incident radiation and the size of the domain $V$. For the condition of no atmosphere, the BHR becomes directional hemispherical reflectance (DHR). We reserve symbol $A_0$ to denote its value.

**Problem 3.1:** Derive the BRF and DHR for a mirror.

Single scattering albedo, $\omega(r, \Omega')$ (see also Section I.2), at the spatial point $r$ and in the direction $\Omega'$ is the probability that photon travelling along the direction $\Omega'$ will be scattered by the point $r$. This probability is the ratio of the scattering coefficient to the extinction coefficient (See sections II.3 and II.4); that is,

$$\omega(r, \Omega') = \frac{\sigma_r'(r, \Omega')}{\sigma(r, \Omega')} = \frac{u_\perp(r)}{u_\perp(r)} \frac{\pi^{-1} \int_{2\pi} \Gamma(r, \Omega' \rightarrow \Omega) \, d\Omega}{G(r, \Omega')} \quad (3.5)$$
If \( u_L(r) \neq 0 \) at \( r \), one can cancel \( u_L \) in Eq. (3.5). The condition \( u_L(r) = 0 \) means that there are no phytoelements in an elementary volume about the point \( r \) and thus the probability of the scattering event is zero, i.e., \( \omega(r, \Omega) = 0 \). Single scattering albedo can take values between 0 and 1. The vegetation canopy is said to be absorbing medium if \( \omega(r, \Omega) = 0 \) at any spatial point \( r \in V \) for which \( u_L(r) \neq 0 \) and in any direction \( \Omega \).

**Problem 3.2:** Explain why the condition “for which \( u_L(r) \neq 0 \)” can not be omitted in the definition of the absorbing medium.

**Problem 3.3:** Show that single scattering albedo satisfies the inequality \( \omega(r, \Omega) \leq \omega \). Here

\[
\sigma = \sup_{r \in V; \Omega, \Omega_L} \left\{ \rho_L^\dagger(r, \Omega', \Omega_L, \Omega) + \tau_L(r, \Omega', \Omega_L); \quad \rho_L^\ast(r, \Omega', \Omega_L) + \tau_L^\ast(r, \Omega', \Omega_L) \right\},
\]

where \( \rho^L, \rho^\ast, \tau^L \) and \( \tau^\ast \) are defined in Eq. (2.7). It follows from definition that \( \omega(r, \Omega) \leq 1 \). Can \( \omega_0 \) take on a value which exceeds 1? Find \( \omega_0 \) for the purely absorbing medium.

**Problem 3.4:** Let \( \omega_0(r, \Omega) \) be

\[
\omega_0(r, \Omega) = \frac{u_L(r)}{u_L(r)} \left( \frac{1}{\pi} \int \Gamma(r, \Omega' \rightarrow \Omega) \, d\Omega' \right) G(r, \Omega).
\]

Note that the integration of \( \Gamma \) is performed over incident directions. Prove the equality \( \omega_0(r, \Omega) = \omega(r, \Omega) \) for the bi–Lambertian phase function (see section II.4). Show that this property loses its validity in the general case of leaf normal distribution.

**Problem 3.5:** Let a vegetation canopy located in the parallelepiped \( V \) is isotropically illuminated from above and bounded from below and lateral sides by a black surface, i.e., \( B(r_B, \Omega) = 1/\pi \) if \( r_B \in \delta V \) and \( B(r_B, \Omega) = 0 \), otherwise. Prove that the BHR is less than 1. Use Theorem I.1.

**Problem 3.6:** Prove that the BHR is an increasing function with respect to single scattering albedo. Do not use the assumptions of the Problem 3.5. Use Theorem I.1.

### III.3. Boundary Conditions

The boundary conditions for a three-dimensional canopy are also three-dimensional. Indeed, the radiation entering the canopy through the top, \( \delta V_t \), through the bottom, \( \delta V_b \), and through the lateral, \( \delta V_l \), surfaces are different. Therefore we consider a very general form of boundary conditions, namely,

\[
I(r_B, \Omega) = \frac{1}{\pi} \int_{\Omega \cdot n(r_B) > 0} \rho_B(r_B', \Omega; r_B, \Omega) \cdot n(r_B') \cdot \Omega' \cdot I(r_B', \Omega') \, d\Omega' + q(r_B, \Omega), \quad n(r_B) \cdot \Omega < 0. \tag{3.8}
\]

Here \( r_B \) and \( r_B' \) are points on the canopy boundary \( \delta V \); \( n(r_B) \) is the outward normal at the point \( r_B \); \( \rho_B(r_B', \Omega; r_B, \Omega) \) is the boundary scattering function; that is, the probability density that a photon having escaped from the canopy through the point \( r_B' \in \delta V \) and in the direction \( \Omega' \) will come back to it through the point \( r_B \in \delta V \) and in the direction \( \Omega \); and \( q_B(r_B, \Omega) \) is a photon source.
at the canopy boundary $\delta V$. Both $\rho_B$ and $q_B$ are wavelength dependent. The boundary scattering function is said to be symmetric if
$$\rho_B(r'_B, \Omega'; r_B, \Omega) = \rho_B(r_B, \Omega; r'_B, \Omega')$$
The magnitude of boundary scattering is described using the following variable [Germogenova, 1986]
$$\rho_{B,V} = \sup_{r'_B \in V, \Omega \cdot n(r'_B) > 0} \frac{1}{\pi} \int_{\Omega \cdot n(r'_B) > 0} \rho_B(r'_B, \Omega'; r_B, \Omega) | n(r_B) \cdot \Omega | d\Omega.$$  (3.9)

Note that the integration of $\rho_B$ is performed over scattering phase points $(r_B, \Omega)$. The radiative transfer problem can now be formulated as follows: find the intensity $I(r, \Omega)$ which satisfies the transport equation (3.1) within the domain $V$ and the conditions (3.8) on the canopy boundary $\delta V$. The variables $\varpi$, $\tau_0(V)$ introduced in Section 1.8 and $\rho_{B,V}$ are basic characteristics of leaf optical properties, canopy structure and canopy-boundary interaction. It follows from the uniqueness theorem that the conditions $\varpi \leq 1$, $\rho_{B,V} < 1$ and $\tau_0(V) < \infty$ guarantee the existence and uniqueness of the solution to the boundary value problem given by (3.1) and (3.8).

Consider the sample stand depicted as a square in Figure 2.1. For a remote sensing perspective, a quantitative description of radiation reflected by a pixel of given dimension (e.g., the square in Figure 2.1) is needed to interpret remotely sensed surface reflectances. Let us specify boundary conditions for our sample stand [Knyazikhin et al., 1997].

**Canopy upper boundary.** The upper boundary of the sample stand adjoins the atmosphere. Therefore radiation penetrating into the canopy through the upper boundary $\delta V_t$ is determined by atmospheric conditions, i.e., the upper canopy boundary is exposed both to direct solar irradiance and to diffuse radiation from all points of the sky. The former is caused by photons in the solar parallel beam which arrive at the upper canopy boundary without experiencing a collision. The latter results from photon-atmosphere interactions. Thus, boundary conditions at the upper boundary can be written as
$$I(r_t, \Omega) = c_T(r_t) \delta(\Omega - \Omega_0) + d(r_t, \Omega), \quad r_t \in \delta V_t; \quad \Omega \cdot n(r_t) < 0.$$  (3.10)

Here $r_t$ denotes points on the upper boundary $\delta V_t$; $\delta(\Omega - \Omega_0)$ is the Dirac delta function; $\Omega_0 \sim (\mu_0, \phi_0)$ is the unit vector directed toward the Sun; $c_T(r_t)$ is the intensity of direct solar radiation, $d(r_t, \Omega)$ is the intensity of diffuse radiation; and $n(r_t)$ is the outward normal at the point $r_t$. Both $c_T$ and $d(r_t, \Omega)$ are wavelength dependent. In terms of notations used in (3.8), $\rho_B(r'_B, \Omega'; r_B, \Omega) = 0$, $q(r_B, \Omega) = c_T(r_B) \delta(\Omega - \Omega_0) + d(r_B, \Omega)$ if $r_B$ belongs to $\delta V_t$.

Let $F_z(r_t)$ be downward horizontal radiative flux density (see section I.1) at the point $r_t$; that is,
$$F_z(r_t) = \int_{\Omega | n(r_t) \cdot \Omega | > 0} I(r_t, \Omega) | n(r_t) \cdot \Omega | d\Omega = F_{z,dir}(r_t) + F_{z,diff}(r_t), \quad r_t \in \delta V_t.$$  (3.11)

Here $F_{z,dir} = c_T(r_t) |\mu_0|$ and
\[
F_{z,\text{diff}}(r_t) = \int_{2\pi} d(r_t, \Omega) |\Omega \cdot n(r_t)| d\Omega, \tag{3.12}
\]

are downward horizontal flux densities of direct and diffuse radiation, respectively. The ratio \(f_{\text{dir}}\), of direct radiation to the total radiation incident on the canopy is defined as

\[
f_{\text{dir}}(r_t) = \frac{F_{z,\text{dir}}(r_t)}{F_{z,\text{dir}}(r_t) + F_{z,\text{diff}}(r_t)}, \quad r_t \in \delta V_t. \tag{3.13}
\]

It is conventional to parameterize the boundary conditions at the upper canopy boundary in terms of the ratio \(f_{\text{dir}}\), the downward horizontal flux density \(F_z(r_t)\), and anisotropy of diffuse radiation \(d_0(r_t, \Omega) = d(r_t, \Omega)/F_{z,\text{diff}}(r_t)\); that is,

\[
I(r_t, \Omega) = \left[ f_{\text{dir}}|\mu_0|^{-1} \delta(\Omega-\Omega_0) + (1 - f_{\text{dir}})d_0(r_t, \Omega) \right] F_z(r_t), \quad n(r_t) \cdot \Omega < 0. \tag{3.14}
\]

The variables \(f_{\text{dir}}\) and \(F_z(r_t)\) depend on the wavelength. They can be sensed remotely [Diner et al., 1999a] and thus can be taken as input for the evaluation of canopy radiation regime. The anisotropy of diffuse radiation can be assumed wavelength independent. A model of clear–sky radiance proposed by Pokrowski [1929]

\[
d_0(r_t, \Omega) = \left[ 1 - \exp \left( \frac{-0.32}{|\mu|} \right) \right] \frac{1+\Omega \cdot \Omega_0}{1-\Omega \cdot \Omega_0}, \quad \Omega \cdot n(r_t) < 0,
\]

is an example of the angular distribution of incoming diffuse radiation. In the case of the standard overcast sky \((f_{\text{dir}} = 0)\), the intensity of the incoming diffuse radiation in the photosynthetically active region of solar spectrum, 400–700 nm, can be approximated by

\[
d(r_t, \Omega) = i(\pi) \frac{1+b|\Omega \cdot n(r_t)|}{1+b}, \quad \Omega \cdot n(r_t) < 0,
\]

where \(1+b\) is the ratio between sky brightness in the zenith, \(i(\pi)\), and at the horizon, \(i(\pi/2)\) and it varies between 2.1 and 2.4 [Monteith and Unsworth, 1990]. Substituting the above equation into \(d_0(r_t, \Omega) = d(r_t, \Omega)/F_{z,\text{diff}}(r_t)\) and taking into account Eq. (3.12) one can express \(i(\pi)\) and \(d_0\) as

\[
i(\pi) = F_z(r_t) \frac{1+b}{\pi(1+\frac{2}{3}b)}, \quad d_0(r_t, \Omega) = F_z(r_t) \frac{1+b|\Omega \cdot n(r_t)|}{\pi(1+\frac{2}{3}b)}. \tag{3.15}
\]

**Problem 3.10:** Derive equations (3.15). Explain why equation (3.15) are not used in remote sensing of vegetation.

The HDRF and BHR defined in (3.3) and (3.4), respectively, are expressed in terms of solution of the transport equation with the upper boundary condition (3.14) and \(f_{\text{dir}} \neq 0\). If \(f_{\text{dir}} = 0\), the HDRF and BHR become the BRF and DHR, respectively.
Canopy bottom boundary. At the bottom of the canopy, a fraction of the radiation can be reflected back into the canopy by the ground. In the remote sensing problems, the boundary scattering function at the canopy bottom can be approximated by $\rho_b(r_b,\Omega; r_b,\Omega_0) = \rho_b(r_b,\Omega \rightarrow \Omega_0) \delta(\Omega - \Omega_0)$, where $r_b, r'_b \in \delta \Omega_b$. The canopy bottom boundary does not emit the radiation at solar wavelength, i.e., $q_b(r_b,\Omega) = 0$. Substituting these equation into (3.8) results in

$$I(r_b,\Omega) = \frac{1}{\pi} \int_{2\pi} \rho_b(r_b,\Omega' \rightarrow \Omega) n(r_b) \cdot \Omega' I(r_b,\Omega') d\Omega', \quad n(r_b) \cdot \Omega < 0.$$  (3.16)

Radiation penetrating through lateral sides. The radiation penetrating through the lateral canopy sides strongly depends on the neighbouring environment. Its influence on the radiative field in the vegetation canopy is essential near the lateral canopy boundary $\delta \Omega_l$. Inaccuracies in the lateral boundary conditions, therefore, cause essential distortions in the simulated radiative field in $\Omega$. These distortions, however, decrease with moving away from this boundary towards the centre of the stand. When the radiative regime in an extended vegetation canopy is analysed, this side effect can be neglected. Inaccuracies in the boundary conditions can, however, put problems in investigating the radiation distribution in a small canopy because the dimension of the “distorted area” can be comparable to the dimension of the pixel. A problem then arises as to how the inaccuracies in the boundary conditions can be minimized. It can be done as follows [Knyazikhin et al., 1997].

We predict the radiative field in a vegetation canopy by solving a one-dimensional transport equation first (e.g., using algorithm by Shabanov et al., [2000]). Its solution, i.e. the vertical profile of the horizontally averaged radiation intensity, is taken then as the radiation penetrating through the lateral canopy boundary which, on average, consider features of both the simulated vegetation canopy and its neighbouring environment. This “predictor–corrector” technique allows us to simulate the lateral boundary conditions, accounting for a complicated process of photon interactions both with the chosen sample stand and with its neighbouring environment. A dependence of size of the “distorted area” induced by utilising this approach on adjoint vegetation, on the atmospheric conditions and the resolution was studied by Kranigk [1996]. In particular, it has been shown that the “distorted area” consists of space points being less than about 5m apart from the lateral boundary of a forest. Thus, the lateral boundary conditions can be expressed as

$$I(r_1,\Omega) = c_i(z_1) \delta(\Omega - \Omega_0) + i_1(z_1,\Omega), \quad r_1 = (x_1, y_1, z_1) \in \delta \Omega_1; \quad \Omega \cdot n(r_1) < 0.$$  (3.17)

Here $c_i(\delta(\Omega - \Omega_0))$ and $i_1$ are wavelength dependent mean intensities of the direct and diffuse radiation at the depth $z$; their sum is the solution to the one–dimensional transport equation. In terms of notations used in (3.8), $\rho_b(r'_b,\Omega'; r_b,\Omega_0) = 0$, $q(r_b,\Omega) = c_i(z_1) \delta(\Omega - \Omega_0) + i_1(z_1,\Omega)$ if $r_b$ belongs to $\delta \Omega_b$. In the case of coarse resolution canopy reflectances, the intensity of radiation penetrating into the canopy through the lateral surface $\delta \Omega_1$ can be set to zero, i.e., $q(r_b,\Omega) = 0$.

Problem 3.11: Show that $\rho_{b,V}$ for the boundary conditions (3.10), (3.14) and (3.17) is given by
III.4. Green’s Function Formalism

The Green’s function concept have been developed in neutron transport several decades ago [Case and Zweifel, 1967; Bell and Glasstone, 1970;]. This is not merely yet another method of solving the radiative transfer equation, but a method to reformulate the radiative transfer problem in terms of some “basic” subproblems and to express the solution to the transport equation with arbitrary sources and boundary conditions as a superposition of the solutions of the basic subproblems. By other words, this technique allows us to split a complicated three-dimensional radiative transfer problem into simpler subproblems which can be evaluated by using existing canopy radiation models.

Let $V$ be the domain bounded by a non-reflecting surface $\delta V$. The volume Green’s function, $G_V(r, \Omega; r', \Omega')$, is the solution of the transport equation corresponding to a unidirectional point source $Q_0(r, \Omega) = \delta(\Omega-\Omega')\delta_V(r-r')$ located at the point $r'$, with zero incoming flux ($q=0; \rho_B=0$); that is,

$$\Omega \cdot \nabla G(r, \Omega; r', \Omega') + \sigma(r, \Omega)G(r, \Omega; r', \Omega') = \int \sigma_s(r, \Omega'' \rightarrow \Omega)G(r, \Omega''; r', \Omega')d\Omega'' + \delta(\Omega-\Omega')\delta_r(r-r'),$$

$G(r_B, \Omega; r', \Omega') = 0, \quad r_B \in \delta V, \quad \Omega \cdot n(r_B) < 0.$

(3.19)

Here $\sigma$ and $\sigma_s$ are the total interaction coefficient (2.13) and the differential scattering coefficient (2.17); $\delta(\Omega-\Omega')$ and $\delta_V(r-r')$ are Dirac delta functions defined as

$$\int f(r)\delta(r-r')dr = f(r'), \quad \int g(\Omega(\Omega')d\Omega') = g(\Omega').$$

Note that the point $r'$ and the direction of the monodirectional source $\Omega'$ are parameters in the boundary value problem (3.19); that is, determination of the Green’s function requires a solution of the radiative transfer problem for each spatial point $r'$ and direction $\Omega'$.

The surface Green’s function, $G_S(r, \Omega; r_B', \Omega')$, is the solution to the transport equation with the source $Q_0(r, \Omega) = 0$ and the boundary conditions

$$G_S(r_B, \Omega; r_B', \Omega') = \delta(\Omega-\Omega')\delta(r_B-r_B'), \quad r_B' \in \delta V, \quad \Omega' \cdot n(r_B') < 0.$$

(3.20)

One can show that [Case and Zweifel, 1967] that $G_S(r, \Omega; r_B', \Omega')$ may be also defined to be solution to the transport equation with zero incident flux, and with a surface source $q_S=\delta_S(r_B-r_B')\delta(\Omega-\Omega')\Omega' \cdot n(r_B')$. Here $\delta_S(r_B-r_B')$ is a two-dimensional delta function (or simple layer over a surface $S$) defined as

$$\rho_{B,V} = \sup_{r' \in \delta V, \Omega \cdot n(r') > 0} \frac{1}{\pi} \int \rho_b(r', \Omega' \rightarrow \Omega)|\Omega \cdot n(r')|d\Omega'$$

This is albedo of the surface underneath the canopy.
\[ \int_V f(r) \delta_S (r - r') \, dr' = \int_S f(r) \delta_S (r - r') \, dS = f(r'), \quad r' \in S. \]

In terms of these two Green’s functions we may write the general solution to the transport equation with arbitrary source \( Q(r, \Omega) \) within \( V \) and arbitrary source \( q(r_B, \Omega) \) on the non-reflecting \((\rho_{B,V} = 0)\) boundary \( \delta V \), namely,

\[
I(r, \Omega) = \int_V \int_0^{4\pi} G(r, \Omega; r', \Omega') Q(r', \Omega') \, dr' \, d\Omega' + \int_{\delta V} d\Omega' G_S (r, \Omega; r_B', \Omega') q(r_B', \Omega'). \tag{3.21}
\]

The first term in (3.21) is the solution of the transport equation with the internal source \( Q(r, \Omega) \) and with zero incoming flux. The proof of this property follows directly from multiplying the equation (3.19) by \( Q(r', \Omega') \) and integrating this equation over \( r' \) and \( \Omega' \). The second term describes the radiative field in three-dimensional medium generated by the boundary source \( q(r_B, \Omega) \). This statement can be proved in a similar fashion. It follows from the linearity of the transport equation that the sum of these components satisfies the transport equation with the internal source \( Q(r, \Omega) \) within \( V \) and the source \( q(r_B, \Omega) \) on the non-reflecting \((\rho_{B,V} = 0)\) boundary \( \delta V \).

It is conventional to express the boundary value problem in terms of operator notations [Vladimirov, 1963]. We introduce the differential, \( L \), and integral, \( S \), operators as

\[
LI = \Omega \cdot V I(r, \Omega) + G(r, \Omega) u_L (r, \Omega) I(r, \Omega)
= \frac{dI(r_B + \xi \Omega, \Omega)}{d\xi} + G(r_B + \xi \Omega, \Omega) u_L (r_B + \xi \Omega) I(r_B + \xi \Omega, \Omega), \quad r_B \in \delta V, \quad \Omega \cdot n(r_B) < 0, \tag{3.22}
\]

\[
SI = \frac{u_L (r)}{\pi} \int_0^{4\pi} [\Gamma (r, \Omega' \rightarrow \Omega) I(r, \Omega') \, d\Omega'. \tag{3.23}
\]

In equation (3.22), we represent the spatial point \( r \) as \( r = r_B + \xi \Omega \). Here the point \( r_B \) belongs to the boundary \( \delta V \) and \( \Omega \cdot n(r_B) < 0 \); \( \xi \) denotes the distance between the point \( r \) and the boundary \( \delta V \) along the direction \(-\Omega\). To describe the boundary condition (3.8), a scattering operator defined on the boundary \( \delta V \) is introduced as

\[
R^+ = \frac{1}{\pi} \int_{\delta V} [\rho_B (r_B', \Omega; r_B, \Omega) \mid n(r_B') \cdot \Omega'] I(r_B', \Omega') \, d\Omega'. \tag{3.24}
\]

Here use the notation \( I^-(r_B, \Omega) \) to denote the intensity of radiation entering into the domain \( V \), i.e., \( r_B \in V \) and \( n(r_B) \cdot \Omega < 0 \), and \( I^+(r_B, \Omega) \) – the intensity of canopy leaving radiation, i.e., \( r_B \in V \) and \( n(r_B) \cdot \Omega > 0 \). The operator \( R \) is defined for \( I^+ \). Using this notation, the boundary value problem (3.1), (3.8) can be expressed as

\[
LI = SI, \quad I^- = R I^+ + q. \tag{3.25}
\]
Here $q(r_B, \Omega)$ is a boundary source defined on the canopy boundary in the incoming directions, i.e., $r_B \in V$ and $n(r_B) \cdot \Omega < 0$. Throughout this section we will use the notations $r_t$, $r_l$ and $r_b$ to denote points on the top ($\delta V_t$), lateral ($\delta V_l$), and bottom ($\delta V_b$) boundaries.

Consider the boundary conditions (3.10), (3.16) and (3.17) with $c_l = 0$ and $i_1 = 0$; that is,

\[ I^-(r_t, \Omega) = c_T \delta(\Omega - \Omega_0) + d(r_t, \Omega), \quad I^-(r_l, \Omega) = 0, \quad I^-(r_b, \Omega) = R I^+ . \]  \hspace{1cm} (3.26)

The solution of the transport equation $L I = S I$ with boundary condition (3.26) can be represented by the sum of two components, viz., $I = I_0 + I_{\text{diff}}$ where $I_0$ is the incident direct radiation that has not undergone interactions in the canopy, and $I_{\text{diff}}$ is the intensity of photons scattered one or more times in the canopy (the diffuse component). The first component $I_0$ satisfies the following boundary value problem

\[ L I_0 = 0, \quad I_0^-(r_t, \Omega) = c_T \delta(\Omega - \Omega_0), \quad I_0^-(r_l, \Omega) = 0, \quad I_0^-(r_b, \Omega) = 0 . \]  \hspace{1cm} (3.27)

The intensity of the diffuse radiative field is a solution of the following operator equation

\[ L I_{\text{diff}} = S I_{\text{diff}} + S I_0, \quad I_{\text{diff}}^-(r_t, \Omega) = d(r_t, \Omega), \quad I_{\text{diff}}^-(r_l, \Omega) = 0, \quad I_{\text{diff}}^-(r_b, \Omega) = R I_{\text{diff}}^+ + R I_0^+ . \]  \hspace{1cm} (3.28)

**Problem 3.12:** Show that $I_0$ is given by

\[ I_0(r, \Omega) = c_T \pi \xi (r, \Omega) H(\Omega - \Omega_0) \exp(-\tau(r, r - \xi_B \Omega_0, \Omega_0) \delta \delta - \Omega_0) H_1(r - \xi_B \Omega_0) \]  \hspace{1cm} (3.29)

Here $\tau(r, r')$ is defined in (2.25), and $\xi_B$ is the distance between $r$ and the boundary $\delta V$ along the direction $-\Omega$, and $H(r)$ is the indicator function of upper boundary whose value is 1 if $r \in \delta V_t$ and zero otherwise.

**Problem 3.13:** Show that $S I_0$ is given by

\[ S I_0 = \frac{c_T \pi}{\xi} \xi (r, \Omega) H(\Omega - \Omega_0) \exp(-\tau(r, r - \xi_B \Omega_0, \Omega_0) \delta \delta - \Omega_0) H_1(r - \xi_B \Omega_0) . \]  \hspace{1cm} (3.30)

**Problem 3.14:** Show that $R I_0^+$ is given by

\[ R I_0^+ = \frac{c_T \rho_B (r_b, \Omega_0 \rightarrow \Omega) \exp(-\tau(r, r - \xi_B \Omega_0, \Omega_0) \delta \delta - \Omega_0) H_1(r - \xi_B \Omega_0) . \]  \hspace{1cm} (3.31)

Because $I_0 = 0$ for the upward directions, the diffuse component $I_{\text{diff}}$ must be specified to evaluate the HDRF (3.3), i.e.,

\[ R(\Omega, \Omega_0) = \frac{\langle I_{\text{diff}}(\Omega) \rangle}{\pi^{-1} \langle F_z, \text{dir} \rangle} , \]  \hspace{1cm} (3.32)

**Problem 3.15:** Show that HDRF can be represented as

\[ R(\Omega, \Omega_0, f_{\text{dir}}) = f_{\text{dir}} \pi \mu_{I_0}^{-1} \langle I_{\text{diff},0} \rangle + (1 - f_{\text{dir}}) \pi \langle I_{\text{diff},0} \rangle . \]  \hspace{1cm} (3.33)

Here $I_{\text{diff},0}$ satisfies (3.28) with $d(r, \Omega) = 0$ and $c_T = 1$, and $I_{\text{diff},0}$ is a solution of (3.28) with $d(r, \Omega) = d_0(r, \Omega)$ and $c_T = 0$. Assume that the flux $F_z(r_t)$ defined in (3.11) does not depend on $r_t$. Find expression for BRF.
Problem 3.16: Let $R = 0$. Show that HDRF can be represented as

\[
R(\Omega, \Omega_0, f_{\text{dir}}) = f_{\text{dir}} \pi \int_{\delta V_t} |\mu_0|^{-1} \left\langle G_s(r_t, \Omega; r'_t, \Omega_0) \right\rangle \text{d}r'_t + (1 - f_{\text{dir}}) \pi \int_{\Omega} \int_{\Omega'} \left\langle G_s(r_t, \Omega; r'_t, \Omega') \right\rangle \text{d}d_0(r'_t, \Omega') \text{d}\Omega'.
\] (3.34)

Assume that the flux $F_z(r_t)$ defined in (3.11) does not depend on $r_t$. Find expression for BHR and DHR.

Problem 3.17: Consider the boundary value problem $L I = Q$, $I^- = 0$. Find the operator $L^{-1}$, i.e., find a solution $I = L^{-1} Q$ to this boundary value problem.

Problem 3.18: Consider the boundary value problem $L I = S I + Q$, $I^- = 0$. Show that $I = (E - L^{-1} S)^{-1} L^{-1} Q$ is the solution of this boundary value problem.

III.5. Decomposition of the solution to the boundary value problem

The Green’s function allows us to express the solution to the transport equation with arbitrary sources and non-reflecting ($\rho_{B,V} = 0$) boundary conditions as a superposition of the solutions of the basic subproblems. Now we investigate the case when the lower canopy boundary $\delta V_t$ can reflect the radiation. Consider the transport equation the transport equation $L I = S I$ with boundary condition (3.26).

It follows from the linearity of the transport equation $L I = S I$ that its solution can be represented as the sum

\[
I(r, \Omega) = I_{\text{bs}}(r, \Omega) + I_{\text{rest}}(r, \Omega).
\] (3.35)

Here $I_{\text{bs}}$ is the solution of the “black-soil problem” which satisfies the boundary value problem,

\[
L I_{\text{bs}} = S I_{\text{bs}}, \quad I_{\text{bs}}(r_t, \Omega) = c_T \delta(\Omega - \Omega_0) + d(r_t, \Omega), \quad I_{\text{bs}}(r_l, \Omega) = 0, \quad I_{\text{bs}}(r_b, \Omega) = 0.
\] (3.36)

The function $I_{\text{rest}}$ satisfies equation $L I_{\text{rest}} = S I_{\text{rest}}$ and boundary conditions expressed as

\[
I_{\text{rest}}(r_t, \Omega) = 0, \quad I_{\text{rest}}(r_l, \Omega) = 0, \quad I_{\text{rest}}(r_b, \Omega) = R I^+.
\]

Note that $I_{\text{rest}}$ depends on the solution of the “complete transport problem,” i.e., $I = I_{\text{bs}} + I_{\text{rest}}$. It describes radiative field due to the interaction between the underlying surface and the vegetation canopy. The lower boundary conditions can be rewritten as

\[
I_{\text{rest}}(r_b, \Omega) = R I^+ T.
\] (3.37)

Here $T$ is downward radiation flux density, i.e.,

\[
T(r_b) = \int_{n(r_b) \cdot n(r_b) > 0} I(r_b, \Omega') \text{d}\Omega'.
\] (3.38)

To parameterize the contribution of the surface underneath the canopy (soil and/or understory) to the canopy radiation regime, an effective ground reflectance is introduced, namely,
\[
\rho_{\text{eff}}(r_b) = \frac{\pi^{-1} \int_{n(r_b) \cdot \Omega > 0} \Omega \cdot n(r_b) |l(r_b, \Omega')\rangle d\Omega}{\int_{n(r_b) \cdot \Omega > 0} \Omega \cdot n(r_b) |l(r_b, \Omega')\rangle d\Omega}. \quad (3.39)
\]

Here \( \rho_b \) is the bidirectional reflectance factor of the canopy bottom introduced in (3.16).

**Problem 3.19:** Show that \( \rho_{\text{eff}}(r_b) \leq \rho_{V,B} \).

**Problem 3.20:** Show that
\[
\inf_{n(r_b) \cdot \Omega > 0} \frac{1}{\pi} \int \rho_b (r_b, \Omega' \rightarrow \Omega) |\Omega \cdot n(r_b)\rangle d\Omega \leq \rho_{\text{eff}}(r_b) \leq \sup_{n(r_b) \cdot \Omega > 0} \frac{1}{\pi} \int \rho_b (r_b, \Omega' \rightarrow \Omega) |\Omega \cdot n(r_b)\rangle d\Omega.
\]

**Problem 3.21:** Find the effective ground reflectance for the case of Lambertian surface underneath the canopy.

Note that the effective ground reflectance depends on the radiation regime in the vegetation canopy. It follows from Problems 3.19 and 3.10 that the range of variations depends on the integrated bidirectional factor of the ground surface only.

Consider the ratio \( RI^+/T \). We have
\[
\frac{RI^+}{T} = \rho_{\text{eff}} \frac{RI^+}{\rho_{\text{eff}} T} = \rho_{\text{eff}}(r_b) d_b(r_b, \Omega).
\]

Here \( d_b \) is the effective ground anisotropy defined as
\[
d_b(r_b, \Omega) = \frac{RI^+}{\rho_{\text{eff}} T} = \frac{1}{\rho_{\text{eff}}(r_b)} \cdot \int_{2\pi} \Omega' \cdot n(r_b) |l(r_b, \Omega')\rangle d\Omega'. \quad (3.40)
\]

**Problem 3.22:** Show that
\[
\int_{2\pi} d_b(r_b, \Omega) |\Omega \cdot n(r_b)\rangle d\Omega = 1.
\]

**Problem 3.23:** Find the effective ground anisotropy for the case of Lambertian surface underneath the canopy.

Thus, the lower boundary conditions (3.37) can be rewritten as
\[
\Gamma_{\text{res}}(r_b, \Omega) = \rho_{\text{eff}}(r_b) d_b(r_b, \Omega) T(r_b). \quad (3.41)
\]

We neglect variations in \( \rho_b \) and \( d_b \) caused by variation in the solution \( I \), i.e., they are determined by ground reflectance properties and independent on the vegetation canopy. Problems 3.19–3.23 provide some arguments confirming this assumption. However, the downward radiation flux density is sensitive to both ground reflectance properties and radiation regime within the
vegetation canopy. This variable must be carefully specified. We use equation (3.21) to express $I_{\text{rest}}$ in terms of the Green’s function, namely,

$$ I_{\text{rest}}(r, \Omega) = \int_{\delta V_b} \int d\Omega' G_S(r, \Omega; r'_b, \Omega') \rho_{\text{eff}}(r'_b) d_b(r'_b, \Omega') T(r'_b). $$

Substituting (3.42) into (3.35) results in

$$ I(r, \Omega) = I_{bs}(r, \Omega) + \int_{\delta V_b} \int d\Omega' G_S(r, \Omega; r'_b, \Omega') \rho_{\text{eff}}(r'_b) d_b(r'_b, \Omega') T(r'_b). $$

Substituting the above equation into (3.38) yields

$$ T(r_b) = T_{bs}(r_b) + \int_{\delta V_b} \rho_{\text{eff}}(r'_b) G_d(r_b, r'_b) T(r'_b) dr'_b. $$

Here $T_{bs}$ is the downward radiation flux density at the lower boundary for the case of a black surface underneath the vegetation canopy, i.e.,

$$ T_{bs}(r_b) = \int_{\delta V_b} I_{bs}(r_b, \Omega') |\Omega' \cdot n(r_b)| d\Omega', $$

the function $G_d$ is an weighted integral of the Green’s function over downward directions, namely,

$$ G_d(r_b, r'_b) = \int_{\delta V_b} \int d\Omega' |\Omega \cdot n(r_b)| \int_{\delta V_b} G_S(r_b, \Omega; r'_b, \Omega') d_b(r'_b, \Omega') d\Omega'. $$

**Problem 3.24:** Show that the function

$$ I_d(r, \Omega) = \int_{\delta V_b} \int d\Omega' G_S(r, \Omega; r'_b, \Omega') d_b(r'_b, \Omega') d\Omega' $$

is the intensity of radiation field in vegetation canopy generated by the anisotropic heterogeneous source $d_b$ located at the canopy bottom. It follows from this property that

$$ \int_{\delta V_b} G_d(r_b, r'_b) dr'_b = \int_{\delta V_b} I_d(r_b, \Omega) |\Omega \cdot n(r_b)| d\Omega. $$

The function $G_d(r_b, r'_b)$ is the downward radiation flux density at $r_b$ generated by the source $d_b$ located at $r_b$. We can resolve equation (3.44) with respect to $T$ and substitute this solution into (3.42). As a result, the three-dimensional field (3.35) is expressed in terms of ground reflectance properties (the effective ground reflectance and effective ground anisotropy) which are independent on the vegetation canopy; the radiation field in the vegetation canopy bounded at the bottom by a black surface (black soil problem) and radiation field (3.47) in the vegetation canopy generated by anisotropic heterogeneous source $d_b$ located at the surface underneath the canopy.
The following estimation of $I_{\text{est}}$ can be performed. Consider the following value

$$A^*(r_b) = \frac{\int G_d(r_b, r'_b) T(r'_b) \, dr'_b}{T(r_b)},$$

(3.49)

which is the bihemispherical reflectance of the vegetation canopy illuminated by an anisotropic source from below. Its variation caused by variation in $T$ is significantly less than variation in $T$. Let $\bar{A}^*$ be the mean $A^*$. Substituting (349) into (3.44) and assuming $\rho_{\text{eff}}$ to be independent on $r_b$, one can rewrite (3.44) in the form

$$T(r_b) = T_{bs}(r_b) + \bar{A}^* \rho_{\text{eff}} T(r_b).$$

We then average this equation over the ground surface. Resolving the averaged equation with respect to the mean $T$, one obtains

$$\bar{T} = \frac{T_{bs}}{1 - \rho_{\text{eff}} \bar{A}^*}.$$  

(3.50)

Here $\bar{T}$ and $T_{bs}$ are mean $T$ and $T_{bs}$, respectively. We replace $T(r_b)$ in (3.43) by their mean $\bar{T}$. Equation (3.43) takes the form

$$I(r, \Omega) \approx I_{bs}(r, \Omega) + \rho_{\text{eff}} \bar{T} \int \delta n_b(r, \Omega') d_r(r_b, \Omega') d\Omega'$$

$$= I_{bs}(r, \Omega) + \frac{\rho_{\text{eff}}}{1 - \rho_{\text{eff}} \bar{A}^*} T_{bs} I_d(r, \Omega)$$

(3.51)

Thus we have parameterized the solution of the transport problem in terms of $\rho_{\text{eff}}$ and solutions of the “black-soil problem,” $I_{bs}$, and “$S$ problem,” $I_d$. The solution of the “black-soil problem” depends on Sun-view geometry, canopy architecture, and spectral properties of the leaves. The "$S$ problem" depends on spectral properties of the leaves and canopy structure only.

**Problem 3.24:** Show that the approximate equality in (3.49) can be replaced by an exact equality in the case of one-dimensional transport equation.

### III.6. Eigenvalue Problem and Canopy Spectral Invariant

The underlying structure of the theory is based on the representation of “states” of a system which are to be represented by (i.e., to be one-to-one correspondence with) vectors in some suitable chosen vector space. For example, the intensity $I(r, \Omega)$ as a function of the spatial point $r$ and direction $\Omega$ can be treated as a vector. The measurements of the attributes of these states (“dynamical variables” or “observables”) are described in terms of operations on these vectors. For example, the amount of energy reflected, $E^+(\delta V)$, or absorbed, $E_a(V)$, by a volume $V$
bounded by a surface $\delta V$ can be expressed as an weighted integral of $I(r,\Omega)$ over spatial points and directions (see section III.4). These operations are assumed to be linear. The operation of an operator on a vector is intended to describe a physical operation (measurement) on the system. Operators $L$, $S$ and $R$ are examples of operators on vectors $I(r,\Omega)$ which describe photon interactions with vegetation medium and its boundary. The relation between the mathematical operation and the physical “measurements” is assumed to be the following: that the result of a “measurement” of a dynamical variable must be an eigenvalue of the linear operator representing that dynamical variable. The state in which the dynamical variable has that value is represented by the corresponding eigenvector. The aim of this section is to introduce the eigenvalues eigenvectors of the transport equation and its application to the estimation of spectral variation of canopy absorptance, transmittance, and reflectance.

Let $V$ be an open region in $\mathbb{R}^3$ where radiative transfer occurs and $\delta V$ be the boundary of $V$. The domain $V$ is variable in this section. An eigenvalue of the transport equation is a number $\theta$ such that there exists a function $\varphi$ which satisfies

$$\theta [\Omega \cdot \nabla \varphi(r,\Omega) + \sigma(r,\Omega)\varphi(r,\Omega)] = \int_{4\pi} \sigma_s(r,\Omega' \rightarrow \Omega)\varphi(r,\Omega')d\Omega'$$

(3.52)

with vacuum boundary conditions, i.e., $\varphi(r_B,\Omega) = 0$ if $r_B \in \delta V$ and $\Omega \cdot n(r_B) < 0$. The function $\varphi(r,\Omega)$ is termed an eigenvector corresponding to the given eigenvalue $\theta$. Under some general conditions [Vladimirov, 1963], the set of eigenvalues $\theta_k$, $k=0,1,2, \ldots$, and eigenvectors $\varphi_k(r,\Omega)$, $k=0,1,2, \ldots$, is a discrete set. The transport equation has a positive eigenvalue which corresponds to a positive eigenvector. This eigenvalue is greater than the absolute magnitudes of the remaining eigenvalues. This means that only one eigenvector, say $\varphi_0$, takes on positive values for any $r$ and $\Omega$.

As it was discussed in section III.3, the transport equation with a reflecting boundary (3.26) can be reduced to the solution of the black–soil problem (3.36) and S problem (3.47). The boundary conditions for the black soil problem are wavelength dependent. However, the incoming radiation can be parameterized in terms of two scalar values: $f_{\text{dir}}$ and total flux $F_z$ of incoming radiation [see section III.2.1, equation (3.14)]. It allows representing the black-soil problem as a sum of two radiation fields. The first is generated by the monodirectional component of solar radiation incident on the top surface of the canopy boundary and, the second, by the diffuse component. Dividing the transport equations and boundary conditions which define these problems by $f_{\text{dir}}F_z$ and $(1-f_{\text{dir}})F_z$ one can reduce them to transport problems with wavelength-independent boundary conditions (see Problem 3.15). The boundary condition of the S problem expressed by equations $I_d = d_b$ (see Problem 3.24) are wavelength independent. Equation (3.51) expresses solution of (3.26) via solutions of the black soil and S problems. Thus the spectral variation of the radiative field in vegetation canopies can be described, when the spectral variation of the solution of the transport equation with wavelength-independent boundary conditions is known. Therefore we consider the transport equation $LI_{\lambda} = S_{\lambda}I_{\lambda}$ with conditions $I_{\lambda}^- = q$ and assume that the source $q$ defined on the boundary $\delta V$ in incoming directions (i.e., $\Omega \cdot n(r_B) < 0$) does not depend on wavelength. Here dependence of $I$ and $S$ on the wavelength $\lambda$ is included in notations.
Differentiating equation $LI_\lambda = S_\lambda I_\lambda$ and boundary conditions $I_\lambda^- = q$ with respect to wavelength $\lambda$, we obtain that the function

$$u_\lambda(r, \Omega) = \frac{dI_\lambda(r, \Omega)}{d\lambda}$$

satisfies the equation

$$Lu_\lambda = \frac{dS_\lambda I_\lambda}{d\lambda}$$

and boundary conditions $u_\lambda = 0$. We expand the solution $I_\lambda$ in eigenvalues,

$$I_\lambda(r, \Omega) = a_0(\lambda)\varphi_0(\lambda, r, \Omega) + \sum_{k=1}^{\infty} a_k(\lambda)\varphi_k(\lambda, r, \Omega),$$

where coefficients $a_k$ do not depend on spatial or angular variables. Here $\varphi_k$ is an eigenvector corresponding to the eigenvalue $\theta_k$, i.e., $\theta_k(\lambda)\varphi_k(\lambda) = S_\lambda \varphi_k(\lambda)$. We separate the positive eigenvector $\varphi_0$ into the first summand. As described above, only this summand, $a_0\varphi_0$, takes on positive values for any $r \in V$ and $\Omega$. Substituting (3.55) into equation (3.54) we get

$$\sum_{k=0}^{\infty} L u_k(\lambda) = \sum_{k=0}^{\infty} \frac{d}{d\lambda} a_k(\lambda) S \varphi_k(\lambda) = \sum_{k=0}^{\infty} \frac{d}{d\lambda} a_k(\lambda) \theta_k(\lambda) L \varphi_k(\lambda) = \sum_{k=0}^{\infty} L \frac{d}{d\lambda} a_k(\lambda) \theta_k(\lambda) \varphi_k(\lambda),$$

where $u_k = d(a_k\varphi_k)/d\lambda$. It follows from the above relationship that

$$\sum_{k=0}^{\infty} L [u_k(\lambda) - \frac{d}{d\lambda} a_k(\lambda) \theta_k(\lambda) \varphi_k(\lambda)] = \sum_{k=0}^{\infty} L \left[ \frac{d}{d\lambda} (1 - \theta_k(\lambda)) a_k(\lambda) \varphi_k(\lambda) \right].$$

Thus,

$$\frac{d}{d\lambda} (1 - \theta_k(\lambda)) a_k(\lambda) \varphi_k(\lambda) = 0.$$

It follows from the above equation that $(1 - \theta_k(\lambda)) a_k(\lambda) \varphi_k(\lambda)$ does not depend on $\lambda$, i.e., $(1 - \theta_k(\lambda)) a_k(\lambda) \varphi_k(\lambda) = (1 - \theta_k(\lambda_0)) a_k(\lambda_0) \varphi_k(\lambda_0)$ where $\lambda_0$ is an arbitrary chosen wavelength. We have,

$$a_k(\lambda) \varphi_k(\lambda) = \frac{1 - \theta_k(\lambda_0)}{1 - \theta_k(\lambda)} a_k(\lambda_0) \varphi_k(\lambda_0).$$
Thus if we know the kth summand of the expansion in equation (3.57) at a wavelength \( \lambda_0 \), we can easily find this summand for any other wavelength.

We introduce \( e \), the monochromatic radiation at wavelength \( \lambda \) intercepted by the vegetation canopy,

\[
e(\lambda) = \int \int dr d\Omega \sigma(r, \Omega) \phi_\lambda(r, \Omega),
\]

and \( e_0 \) as

\[
e_0(\lambda) = \int \int dr d\Omega \sigma(r, \Omega) \phi_\lambda(r, \Omega) \cdot \phi_0(\lambda, r, \Omega) dr d\Omega.
\]

Given \( e \), we can evaluate the fraction \( a \) of radiation absorbed by the vegetation at the wavelength \( \lambda \) as

\[
a(\lambda) = [1 - \omega(\lambda)]e(\lambda),
\]

where \( \omega(\lambda) \) is the leaf albedo. Below an estimation of \( e_0 \) will be performed. This value is close to \( e \). We skip a precise mathematical proof of this fact here. An intuitive explanation is as follows: Putting (3.55) in (3.56) and integrating the series results in only the positive term containing \( a_0 \phi_0 \). As a result, \( e(\lambda)/e(\lambda_0) \approx e_0(\lambda)/e_0(\lambda_0) \). Let us derive the dependence of \( e \) on wavelength. Substituting equation (3.55) into equation (3.57) we obtain

\[
e_0(\lambda) = \frac{1 - \theta_0(\lambda_0)}{1 - \theta_0(\lambda)} e_0(\lambda_0),
\]

where \( \theta_0 \) is the positive eigenvalue corresponding to the positive eigenvector \( \phi_0 \). Taking into account equation (3.58), we can also derive the following estimation for \( a \):

\[
a(\lambda) = \frac{1 - \gamma_0(\lambda_0)}{1 - \gamma_0(\lambda_0)} \cdot \frac{1 - \omega(\lambda)}{1 - \omega(\lambda_0)} a(\lambda_0).
\]

Thus given canopy absorptance at wavelength \( \lambda_0 \), we can evaluate this variable at any other wavelength. Figure 3.3 shows spectral variation of the fraction of energy absorbed by the vegetation canopy \( a \) for uniform and planophile leaves.
A somewhat more complicated technique is realized to derive an approximation for canopy transmittance. For simplicity, specular reflection by leaves is ignored here which is not critical to the following analysis. Under these assumptions, the area scattering phase function can be expressed as

$$\Gamma_{\lambda}(r, \Omega' \rightarrow \Omega) = \frac{1}{2\pi} \int_{\Omega \cdot \Omega_L > 0} \left[ t_{L,\lambda}(r, \Omega, \Omega_L) g_{L}(r, \Omega_L) \left( \Omega \cdot \Omega_L \right) \left( \Omega' \cdot \Omega_L \right) \right] d\Omega_L$$

$$+ \frac{1}{2\pi} \int_{\Omega \cdot \Omega_L < 0} \left[ r_{L,\lambda}(r, \Omega, \Omega_L) g_{L}(r, \Omega_L) \left( \Omega \cdot \Omega_L \right) \left( \Omega' \cdot \Omega_L \right) \right] d\Omega_L .$$

Here $r_{L,\lambda}$ and $t_{L,\lambda}$ are the bi-directional reflectance and transmittance factors of an individual leaf which are supposed Lambertian; that is, $r_{L,\lambda}(r, \Omega', \Omega) = \rho(\lambda, r)$ and $t_{L,\lambda}(r, \Omega', \Omega) = \tau(\lambda, r)$ where $\rho$ and $\tau$ are the leaf hemispherical reflectance and transmittance, respectively. The hemispherical leaf albedo can be expressed as $\omega_{\lambda}(r) = \rho(\lambda, r) + \tau(\lambda, r)$. These variables are measurable parameters and are assumed independent of the spatial variable $r$ and direction $\Omega$. With these assumptions, the area scattering phase function can be expressed as

$$\Gamma_{\lambda}(r, \Omega' \rightarrow \Omega) = \rho \Gamma^+(r, \Omega' \rightarrow \Omega) + \tau \Gamma^-(r, \Omega' \rightarrow \Omega) ,$$

(3.60)

where $\Gamma^+$ and $\Gamma^-$ are wavelength independent functions defined as

$$\Gamma^+(r, \Omega' \rightarrow \Omega) = \frac{1}{2\pi} \int_{\Omega \cdot \Omega_L > 0} g_{L}(r, \Omega_L) \left( \Omega \cdot \Omega_L \right) \left( \Omega' \cdot \Omega_L \right) d\Omega_L .$$

Differentiating equation $L_{\lambda} = S_{\lambda} I_{\lambda}$ and boundary conditions $I_{\lambda} = q$ with respect to $\rho$ and $\tau$ and accounting for (3.60), one can obtain that the function

Figure 3.3. Spectral variation canopy transmittance for uniform leaves evaluated with canopy radiation model (points) and from equation (47) for LAI=1.1 (left) and 4.1 (right).
\[ v(r, \Omega; \rho, \tau) = \rho \frac{\partial I_k(r, \Omega)}{\partial \rho} + \tau \frac{\partial I_k(r, \Omega)}{\partial \tau}, \]

satisfies the equation

\[ \Omega \cdot \nabla v + u_\lambda(r) G(r, \Omega) v = \left[ \rho \frac{\partial}{\partial \rho} + \tau \frac{\partial}{\partial \tau} \right] u_\lambda(r) \left[ \frac{1}{4\pi} \Gamma_k(r, \Omega' \rightarrow \Omega) I_k(r, \Omega') d\Omega' \right], \tag{3.61} \]

and the boundary conditions \( v(r_B, \Omega; \rho, \tau) = 0 \) for incoming directions \( \Omega \cdot n(r_B) < 0. \)

Substituting (3.55) into equation (3.61) and accounting for (13), results in

\[ \begin{aligned} \left[ \Omega \cdot \nabla + \sigma(r, \Omega) \right] \sum_{k=0}^\infty \left\{ \rho \frac{\partial (1-\theta_k) a_k \varphi_k}{\partial \rho} + \tau \frac{\partial (1-\theta_k) a_k \varphi_k}{\partial \tau} \right\} = 0. \end{aligned} \]

Here \( \theta_k(\rho, \tau) \) is the eigenvalue corresponding to the eigenvector \( \varphi_k \). It follows from this equation that

\[ \rho \frac{\partial (1-\theta_k) a_k \varphi_k}{\partial \rho} + \tau \frac{\partial (1-\theta_k) a_k \varphi_k}{\partial \tau} = 0, \quad k = 0, 1, 2, \ldots. \tag{3.62} \]

The general solution of the first-order partial equation (3.62) with respect to the function \( (1-\theta_k) a_k \varphi_k \) can be expressed as [Richards, 1959]

\[ (1-\theta_k(\rho, \tau)) a_k(\rho, \tau) \varphi_k(r, \Omega; \rho, \tau) = f_k(\rho/\tau), \tag{3.63} \]

where \( \rho/\tau=\text{const} \) is the characteristic curve of (3.62), and \( f_k \) is an arbitrary function of one variable. Letting \( x=\rho/\tau \) and \( \omega_0=\rho+\tau \), one can specify this function as

\[ f_k(x) = \left[ 1 - \theta_k(\omega_0, 1, \omega_0, x+1) \right] a_k(\omega_0, 1, \omega_0, x+1) \varphi_k(\omega_0, 1, \omega_0, x+1). \]

Substituting this function into (2.82) one obtains

\[ a_k(\rho, \tau) \varphi_k(\rho, \tau) = \frac{f_k(\rho/\tau)}{1-\theta_k(\rho, \tau)} = \frac{1-\theta_k(\omega_0, 1-\zeta, \omega_0) a_k(\varphi_0, 1-\zeta, \omega_0) \varphi_k(\varphi_0, 1-\zeta, \omega_0)}{1-\theta_k(\rho, \tau)}, \]

where \( \zeta=\tau(\omega_0)/(\tau(\omega_0)+\rho(\omega_0))=\tau(\omega_0)/\omega(\omega_0). \) Thus, if the kth summand of the expansion (3.55) at a reference leaf albedo, \( \omega_0=\omega(\lambda_0) \), for different values of parameter \( \zeta \) is known, the summand for
any other $\omega(\lambda)$ can be easily found. The maximum positive eigenvalue $\theta_0$ corresponding to the positive eigenvector, can be estimated as

$$\theta_0(\rho, \tau) = p\omega_0 = [\rho(\lambda) + \tau(\lambda)]p.$$  \hfill (3.64)

Here $p=1-\exp(-K)$ where $K$ is a wavelength independent constant.

Substituting (3.55) into (3.61) and accounting for (3.63) and (3.64), one obtains an expansion for the canopy transmittance, namely,

$$t(\rho, \tau) = \left\{ \frac{1 - p\omega_0}{1 - p\omega} T_0(r_H; \zeta, \omega_0) + \sum_{k=1}^{\infty} \frac{1 - \theta_k(\omega_0, (1 - \zeta)\omega_0)}{1 - \theta_k(\rho, \tau)} T_k(r_H; \zeta, \omega_0) \right\}_{H}.$$ \hfill (3.65)

Here

$$T_k(r_H; \zeta, \omega_0) = a_k(\omega_0, (1 - \zeta)\omega_0) \int_{\Omega} \varphi_k(r_H, \Omega) d\Omega, \quad k = 0, 1, 2, \cdots$$

Values of $T_k$ depend on the reference hemispherical leaf albedo $\omega_0$, the value of $\zeta$ at the reference leaf albedo, the direction of direct solar radiance $\Omega_0$, and the points $r_H$. Let us consider the first term of the series (3.65) as a function of $\rho(\lambda)$ and $\tau(\lambda)$; that is,

$$T_0(\rho, \tau) = \frac{1 - p\omega_0}{1 - p\omega} T_0(\omega_0, (1 - \zeta)\omega_0)$$

Averaging the canopy transmittance over the angular variable and the ground surface results in the term sub-scripted by 0 to be dominant in the expansion (3.65). However, variations in the summands caused by variation in the direction of incident solar radiation make the coefficient $p$ sensitive to the conditions of canopy illuminations and $\zeta$. Thus, if the canopy transmittance at a reference leaf albedo, $\omega_0=\omega(\lambda_0)$, for different values of parameter $\zeta=\tau(\lambda_0)/\omega(\lambda_0)$ is known, the transmittance for any other $\omega(\lambda)$ can be found.
III.7. Bibliography

Further readings


References